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ABSTRACT

This resource book contains 20 magic tricks based on the properties of whole numbers that are intended to spark the interest and imagination of students. Following each activity, students are asked to write about their discoveries and to create their own magic tricks. A matrix of skills for all the activities and lists of the materials required for each activity are included. Each of the activities has three sections: unraveling the trick, questions to ponder, and further investigations. The emphases of these activities are on student discovery, critical thinking, verbal and written communications, and creativity. The tricks work well in whole class or small group situations. (DDR)

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ALGEBRA MAGIC TRICKS

Algecadabra! Volume 1

Ronald Edwards

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Introduction

This book contains a collection of classic magic tricks based on the properties of whole numbers. Historically these tricks have appeared as recreations or oddities in early mathematics books, magic books, and magazines.

Students find these tricks intriguing, which makes them wonderful motivational activities to begin or end a class. As students do these tricks, they begin to feel that the tricks are indeed "magic," that is, they believe them, but cannot explain them.

Teachers may show their students how to perform a trick, that is, *how it works*. But, this still does not explain *why it works*. When the trick is repeated, students realize there must be a way of predicting the outcome. At this point teachers can lead their students through an algebraic analysis of a trick. In this way, the tricks are the starting point for a mathematical exploration which explains the theory behind the magic.

For example, in one magic trick, a volunteer rolls a pair of dice and uses the resulting top numbers on the dice in a few simple arithmetic computations. The answer to these computations is 49. The answer is 49 every time the trick is done. Students assume that the answer will always be 49 and by induction state this as an unproven fact. The text provides a formal proof of this inductive hypothesis using only elementary algebraic computation.

At first the teacher may lead students through the proofs, but as students become more confident in their use of algebra, they can be given more freedom in the discovery process. Teachers may have students work cooperatively in small groups to explain "algebraically" how a particular trick works. *Unraveling the Trick* and *Questions to Ponder* on the Student Activity pages can be used to guide this exploration. Individuals, small groups, or the class working as a whole may explore the *Further Investigations* following each trick. Hints to these investigations are provided in the *Teacher's Instructions*.

The *Matrix of Skills* on the following page shows the skills necessary to understand each proof and may assist teachers in selecting activities which meet their students' mathematical development or in tieing the explorations to specific topics in an algebra course. First-year algebra students should be capable of understanding all arguments or proofs which are used in this book.

As students become accustomed to using algebra in the proofs, the next step seems natural. They begin to design their own magic tricks using algebra. This last step completes the cycle in the mathematical exploration: *I observe, I conjecture, I form a hypothesis, I construct a proof, I create my own magic tricks.* At this point algebra is no longer simply used to describe and illustrate how a specific trick works, but as a tool in creating new ones. Now students start to think algebraically.

To a great friend and great magician:

Tony Hassini

Matrix of Mathematical Concepts and Skills

Concept or Skill	Activity																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Arithmetic																				
basic computation	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	
square root																		■		
expanded notation										■	■	■	■	■	■	■			■	
multiples	■								■					■	■				■	
factorization													■							
division theorem													■			■	■			
odd and even numbers				■				■			■		■							
prime numbers				■													■			
divisibility properties													■	■	■	■	■			
commutative property	■											■								
distributive property					■			■		■		■	■					■		
permutations											■			■		■				
units of measure										■										
Algebra																				
use of variable(s)	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■
algebraic expressions	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■	■		■
polynomials			■		■	■	■	■	■	■	■	■	■	■	■	■	■	■		
simplifying polynomials		■	■		■	■	■	■	■	■		■		■	■	■	■	■		
adding polynomials			■				■		■	■	■		■	■	■			■		
multiplying polynomials																		■		
special products							■											■		
distributive property			■		■	■		■	■	■		■		■	■	■	■	■		
factoring polynomials											■		■			■		■		
solving equations					■				■											

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NUMBER 1 THE MISSING EIGHT

All conjurers know special properties of the number 12,345,679. Note the absence of an "8." Here is a trick based on properties of "the missing eight."

PERFORMING THE TRICK

1. Write down 12345679.
2. Ask someone which of the digits in the number s/he likes the least and have the person put an "x" over the digit selected.
3. Ask the person to multiply 12345679 by a number you select.
4. All of the digits in this product will match the number marked with the "x."

HOW IT WORKS

Multiply whichever number is marked with the "x" by 9 and use the resulting one- or two-digit number as the multiplier. All the digits in the product will match the digit under the "x."

AN EXAMPLE

If the person indicates a dislike for 7 by putting the "x" above 7, you instruct them to multiply by 9 × 7 or 63.

$$\begin{array}{r} \text{X} \\ 12345679 \\ \times \quad 63 \\ \hline 37037037 \\ +\,740740740 \\ \hline 777777777 \end{array}$$

WHY IT WORKS

Note that 12345679 × 9 = 111,111,111. Now, suppose n represents the digit marked with an "x" in the trick. Verification of the trick is simply based on the multiplication fact above and the associative property of multiplication. When you suggest multiplying by $9n$, the following arithmetic results.

$$\begin{aligned} 12345679 \times 9n &= (12345679 \times 9) \times n \\ &= (111{,}111{,}111) \times n \\ &= nnn{,}nnn{,}nnn. \end{aligned}$$

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An alternative method of verification is to check all of the eight possible cases. That is, you can multiply 12345679 by each of the possible products of 9: $9 \times 1 = 9$, $9 \times 2 = 18$, $9 \times 3 = 27$...$9 \times 7 = 63$, $9 \times 9 = 81$.

This method of verification (checking all possible cases) is called *proof by exhaustion.* It only works when the number of choices is finite which allows one to try each case to exhaust the possibilities. Of course, if the number of possibilities is finite and large, the process can be *exhausting* for the person doing the verification.

Hints to Further Investigations

1. Noting that $123456789 \times 9 = 1111111101$, use an analysis similar to the one used in the original trick.
2. One possibility is writing $12345679 = (1234 \times 10^4) + 5679$.

   Now, $63 \times 12345679 = (63 \times 1234 \times 10^4) + (63 \times 5679) = 777420000 + 357777 = 777777777$.
3. Use a calculator and inductive reasoning.
4. Use a calculator and inductive reasoning.

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THE TRICK: The Missing Eight

1. Write down 12345679.
2. Ask someone which of the digits in the number s/he likes the least and have the person put an "x" over the digit selected.
3. Ask the person to multiply 12345679 by a number that you select.
4. All of the digits in this product will match the number marked with the "x."

UNRAVELING THE TRICK

1. Multiply 12345679 by 9 and observe the result.

2. Choose a variable to represent the number marked with an "x."

3. Write the algebraic expression for 12345679 multiplied by 9 times the number marked with an "x." Use this result to explain the trick.

QUESTIONS TO PONDER

1. Are there any numerical restrictions on the variable chosen?

2. Is the answer predictable if you use the number 123456789?

3. Suppose the subject subtracts the digit selected from 9, multiples this difference by 9 and multiples this result by 12345679. If the final result is announced to you, could you predict which digit was marked with an "x"?

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FURTHER INVESTIGATIONS

1. In THE MISSING 8 trick, determine what happens to the products if 12345679 is replaced by 123456789.

2. Develop a way to use a standard eight-place display calculator to compute the products 12345679 × 63 without causing a calculator overflow.

3. Show that the following rule works for squaring any number made up entirely of 9s.

   For the answer, put down a 1 for the unit's digit; to its left put zeros (one less than the number of 9s in the number you are squaring); next put an 8; and finally, 9s (one less than the number of 9s in the number you are squaring). Thus, $99^2 = 9801$.

4. Multiply 9 by each number in the following sequence: 21, 321, 4321, 54321, 654321, 7654321, 87654321, 987654321. Continue until you discover a pattern in the products.

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NUMBER 2 THE LIGHTNING CALCULATOR

One standard number trick portrays the magician as a *lightning calculator.* There are dozens of variations of this trick and some are suggested as **Further Investigations.**

PERFORMING THE TRICK

1. Choose a volunteer and ask her/him to write down a five-digit number.
2. You write down a number under her/his number.
3. Continue in this manner, alternating numbers, until the column is at least 8 figures long.
4. Have the volunteer find the sum of these numbers.
5. While s/he is summing the column of figures, you write down the sum as fast as you can write the digits.

HOW IT WORKS

After the subject writes a five-digit number, mentally subtract this number from 99999 and write this difference as your number. Suppose this process is repeated two more times creating a column of six numbers. After the subject writes the seventh number, you write any number you wish (it's best to choose a number for which carrying is not involved when adding the seventh and eighth numbers) for the eighth row. To compute the sum of the eight numbers, add the seventh and eighth numbers, subtract 3 from this sum and then place a 3 in front of this result.

AN EXAMPLE

Below, the subject writes the numbers in the odd-numbered rows and you write the numbers in the even-numbered rows.

First row	65421
Second row	34578
Third row	52578
Fourth row	47421
Fifth row	97351
Sixth row	2648
Seventh row	23451
Eighth row	31243

Your lightning calculation of the sum =	300000
	+ 23451
	+ 31243
	− 3
12	354691

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HOW IT WORKS

Suppose the subject writes x five-digit numbers before her/his last number and you follow each of these x numbers with a number that is 99999 minus the subject's choice. The sum of these $2x$ numbers is $x(99999) = x(100000) - x$. Suppose the last two numbers written are y and z. The sum of the complete column of $2x + 2$ numbers will be: $x(100000) - x + (y + z) = x(100000) + (y + z) - x$. Thus, you add the last two numbers $(y + z)$, subtract the number of pairs above the last two rows (x), and then affix the digit x in front of this result.

A VARIATION

A variation of this trick was shown to me by a magician. This variation used four-digit numbers. Before the trick began he already had written the final sum on a piece of paper sealed in an envelope. As the last number was written, he handed me the envelope.

To accomplish this, he simply chose in advance a number for his final entry in the seven-number column, say 3476, and sealed the sum, 33473, in the envelope. After I chose each number, he wrote a number which when added to my number gave a sum of 9999. For example, if I wrote 2487, he wrote 7512. Having done this three times, the sum of these six numbers was 9999 + 9999 + 9999 = 30,000 – 3. He wrote the seventh and last number 3476 (the one he chose before the game started). Now, the sum was 30,000 + 3476 – 3 = 33473, the number sealed in the envelope. After I completed the sum (using a calculator) I opened the envelope and found the same result. Now, that was faster than *lightning calculation*. That was magic.

The magician also told me about a great finale to the trick. Sharpen a piece of soap and use it to write the sum, 33437, on the back of your hand. It will be invisible. Have the subject complete the sum, write it on a piece of paper, and burn the paper. Now you take the ashes and rub them on the back of your soaped hand and the sum will appear.

Hints to Further Investigations

1. Suppose you wish the final sum to be 3581 and you write 584 as the first number on the list. After each selection of a three-digit number by the subject, choose a three-digit number such that the sum of the two is 999. If this is done three times, what is the sum of the seven numbers?
2. You might suggest selecting numbers between 10000 and 90000.
3. After the subject selects each of the first three numbers, you write the arithmetical complement of each number. Will there be a need to subtract 3 from the final sum?

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THE TRICK: The Lightning Calculator

1. Ask someone to write down a five-digit number.
2. You write a number under her/his number.
3. Alternate writing numbers until at least 8 numbers are listed.
4. You give the answer while the other person is still summing the column of numbers.

UNRAVELING THE TRICK

1. Assume the subject selects *n* five-digit numbers and you follow each selection with a number that will make the sum of the two equal 99999. Write an algebraic expression for the sum of the *n* numbers.

2. Now replace 99999 with (100000 – 1) and use the distributive property to remove the parentheses.

3. Choose two different variables to represent the last two numbers selected by the subject and by you.

4. Compute the sum of the expressions in steps #2 and #3 above. Use this sum to explain the trick.

QUESTIONS TO PONDER

1. Are there numerical restrictions on the variable *n* (the number of five-digit numbers selected by the subject)?

2. Are there any restrictions on the last two numbers selected by the subject and you?

3. What would happen if the subject selected 99999 as the first five-digit number? For all your selections to be five-digit numbers, what restrictions must be placed on the subject's selections?

4. How would your prediction be altered if four-digit numbers were selected instead of five-digit numbers?

5. How could you predict the final sum if you omitted selecting your last number?

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FURTHER INVESTIGATIONS

1. Alter the trick so that three-digit numbers are used and so that you will know the sum of the numbers before the subject selects his first number.

2. In the original version of the trick, if someone chose 99999 as a number, this would give away the trick. How can you alter the trick to avoid this situation?

3. In nineteenth century arithmetic texts the arithmetical complement of a number is defined as *the remainder found by subtracting the number from the next highest multiple of 10.* Thus, the arithmetical complement of 8548 is 1452, since 8548 + 1452 = 10000. Alter THE LIGHTNING CALCULATOR trick by incorporating the arithmetical complements of the subject's numbers.

4. Some early arithmetic texts suggested that the arithmetical complement be used in place of subtraction. To subtract two numbers, add the arithmetical complement of the subtrahend to the minuend and then reduce this sum by the multiple of 10 used in obtaining the arithmetical complement. For example,

756		756
− 321	becomes	+ 679
		1435
		− 1000
		435

Compute these differences using "addition of arithmetical complements."

546	3947	5721
− 324	− 1788	− 846

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NUMBER 3 FORETELLING A SUM

In a Fibonacci series each number is the sum of the two preceding ones. This trick, which was shown to me by a mathematics teacher many years ago, is based on such a series. The trick is attributed to the magician Royal Vale Heath.

PERFORMING THE TRICK

1. Choose a volunteer and have her/him write down any two numbers and announce what they are.
2. Have the volunteer find the sum of these two numbers, write it down, and announce the sum.
3. Have the volunteer find the sum of the second and third numbers, write that down, and announce the sum.
4. Give the same instructions for the third and fourth numbers.
5. Instruct the volunteer to continue the pattern until ten numbers are listed.
6. After the seventh number is listed announce: I predict that the sum of these numbers will be _____.
7. Have the volunteer compute the sum of the ten numbers.

HOW IT WORKS

When the subject writes the seventh number on the list, mentally multiply this number by 11 and write the result in plain view. After the list of ten numbers is completed and the sum computed, the sum will match your prediction.

AN EXAMPLE

Suppose the first two numbers selected are 25 and 17. The list of ten numbers with their sum is shown below.

1.	25	
2.	17	
3.	42	
4.	59	
5.	101	
6.	160	
7.	261	*Here you compute 11 × 261 = 2871
8.	421	
9.	682	
10.	1103	
The sum =	2871	which matches your prediction.

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WHY IT WORKS

Let x be the first number chosen and y the second. The list of the ten numbers with their sum is shown below.

1.	x	=	$1x$	
2.	y	=	$1y$	
3.	$x + y$	=	$1x + 1y$	
4.	$[y + (x + y)]$	=	$1x + 2y$	
5.	$[(x + y) + (x + 2y)]$	=	$2x + 3y$	
6.	$[(x + 2y) + (2x + 3y)]$	=	$3x + 5y$	
7.	$[(2x + 3y) + (3x + 5y)]$	=	$5x + 8y$	*$[11 \times (5x + 8y)] = 55x + 88y$
8.	$[(3x + 5y) + (5x + 8y)]$	=	$8x + 13y$	
9.	$[(5x + 8y) + (8x + 13y)]$	=	$13x + 21y$	
10.	$[(8x + 13y) + (13x + 21y)]$	=	$21x + 34y$	
	Sum	=	$55x + 88y$	

*Note that eleven times the seventh entry is $55x + 88y$, the same as the sum. This relationship does not depend upon one's choice of x and y.

Hints to Further Investigations

1. Are there any numerical restrictions on x and y for the algebraic computations performed in the explanation of the trick? If x and y were rational numbers (written as fractions or decimals), would it be easy to mentally multiply the seventh entry by 11?
2. Try several cases and it will become inductively clear that the sum is one less than the number appearing two steps below the line. For a list of twenty Fibonacci numbers this result can be proved by trying all cases, that is, using a proof by exhaustion. If one is familiar with "proof by mathematical induction," the trick can be proved for any number of terms (greater than two) in a Fibonacci series.
3. Try several cases. In each case subtract your sum from the number two steps below the line. What do you observe? Try the same algebraic analysis used in this trick for some specific cases in the series.

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THE TRICK: Foretelling a Sum

1. Write down any two numbers.
2. Find the sum of these two numbers and write it down.
3. Find the sum of the last two numbers (the second number and the sum) and write it down.
4. Continue adding the last two numbers written until ten numbers are listed.
5. The sum of the ten numbers can be predicted from the seventh number in the list.

UNRAVELING THE TRICK

1. Choose two different variables to represent the first and second numbers chosen.

2. Write the algebraic expressions for the other eight numbers in the series using the variables chosen.

3. Sum the ten algebraic expressions.

4. Explain the trick by comparing the computed sum with the seventh algebraic expression.

QUESTIONS TO PONDER

1. Are there any numerical restrictions on the two variables chosen?

2. Could the two numbers chosen both be zero? Could one of them be zero?

3. What positive integers evenly divide the sum of the ten algebraic expressions?

4. What would be the result if the same variable was used for the two chosen numbers?

5. Can you devise a method to multiply 11 times a number so that the prediction step in the trick becomes a quick, mental computation?

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FURTHER INVESTIGATIONS

1. Do x and y have to be whole numbers for this trick to work? Will it work for integer values of x and y? Rational values?

2. In the algebraic solution you derived under ***Unraveling the Trick***, the coefficients of y are: 1, 1, 2, 3, 5, 8, 13, 21, 34. (Notice that each number after the first two is the sum of the two preceding numbers.) This is the beginning of the series which was published by (and named for) the Italian mathematician Leonardo Fibonacci in 1202. List 20 terms of the Fibonacci series in a column. Have someone draw a horizontal line under a number in the series and sum all the numbers above the line. You will immediately be able to predict the sum by making a minor alteration to the number in the series which is two steps below the horizontal line. Try it. What is your prediction? Does it always work?

3. Try the prediction trick you just discovered in #2 above on the series that you used to do the trick. How could your prediction be modified so that the number two steps below the horizontal line allows you to predict the sum above the horizontal line? Can this prediction be generalized so that you can predict the sum for any series where the subject selects the first two numbers? Can you explain algebraically how this prediction works?

4. Locate a reference to *Fibonacci* in a mathematics textbook or reference book and explore other properties of this famous series.

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NUMBER 4 UNVEILING EVEN AND ODD

This trick is presented exactly as it appears in an arithmetic text published in 1760. (Wingate, Edmund. *A Treatise of Common Arithmetic.* London: S. Crowder and Company, 1760. p. 370) You will need two volunteers as the subjects of this trick.

PERFORMING THE TRICK

"Two numbers, the one even and the other odd, being proposed unto two persons, to the end they may (out of your sight) severally chuse one of those numbers; to discover which of these numbers each person hath chosen.

"Suppose you have propounded to Peter and John two numbers, the one even and the other odd, as 10 and 9, and that each of those persons is to chuse one of the said numbers unknown to you...bid Peter multiply that number which he has chosen by 2; and cause John to multiply that number which he has pitched upon by 3; that done, bid them add the two products together, and let them make known the sum to you, or else demand of them whether the said sum be even or odd, or by any other way more secret endeavour to discover it, by bidding them take half of the said sum, for by knowing whether the said sum be even or odd, you obtain the principal end to be aimed at...."

HOW IT WORKS

"...[B]ecause if the said sum be an even number, then infallibly he that multiplied his number by your odd number, (to wit, by 3) did chuse the even number (to wit, 10) but if the said sum happen to be an odd number, then he whom you caused to multiply his number by your odd number (to wit, 3) did infallibly chuse the odd number (to wit, 9)."

AN EXAMPLE

Suppose the two numbers to be chosen are 10 and 9. Also, suppose Peter selects the 10 and John the 9. Peter computes $2 \times 10 = 20$ and John $3 \times 9 = 27$. When you discover the sum of these products ($20 + 27 = 47$) is odd, you will know John (the person who multiplied by 3) selected the odd number, 9.

If Peter selected the 9 and John the 10, however, then the sum of the products, $(2 \times 9) + (3 \times 10) = 48$, would be even and you would know that John (the person who multiplied by 3) selected the even number, 10.

WHY IT WORKS

Verification of this trick is based upon the following properties of whole numbers:

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- An even number plus an odd number is odd;
- An even number plus an even number is even;
- An even number times any number is even;
- An odd number times an odd number is odd.

Let x and y be the two numbers to be chosen, where one is even and the other odd. Suppose the first person chooses x. Since the first person (Peter in the example above) multiplies his number by 2, this product, $2x$, is always even. When the second person (John in the example above) multiples his number by 3, the product, $3y$, will be even if y is even and will be odd if y is odd. If you know that the sum of their two products, $2x + 3y$, is even then you conclude that y must have been even. This tells you that the second person must have chosen the even number. If, however, the sum $2x + 3y$ is odd, then you can conclude that y must have been odd. This tells you that the second person must have chosen the odd number.

	2x	3y	2x + 3y
x = even y = odd	even	odd	odd
x = odd y = even	even	even	even

This trick can also be verified by examining two separate cases: Case (1) the first person selects the odd number, the second person the even number and Case (2) the first person selects the even number and the second person the odd number.

Hints to Further Investigations

1. In Case (1) assume the first person selects the odd number (call it m) and the second person selects the even number (call it n). Show that $2m + 3n$ is even, allowing you to predict that the second person took the even number. In a similar way consider Case (2).
2. You might consider having the subjects divide their sum by 2 and use this quotient to predict which number each person selected.
3. Does the algebraic analysis of the trick specify the odd and even numbers used?
4. Try a method similar to that used in analyzing the original trick.
5. Do not use a fifty-cent piece and a dime for they are both represented by even numbers.
6. Suppose $x = 12$ and $y = 7$ where the prime $p = 2$ divides x. If A selects 12 and multiples by the prime $q = 5$, the result is 60. B computes $7 \times 2 = 14$ and the sum is $60 + 14 = 74$. Since the sum is divisible by p, subject A chose x or 12. Similarly, examine what happens if A chose y.

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THE TRICK: Unveiling Even and Odd

1. The first person chooses either 9 or 10 and multiplies it by 2.
2. The second person takes the other number and multiplies it by 3.
3. They compute the sum of the two products.
4. From this sum you can determine which person chose 9 and which chose 10.

UNRAVELING THE TRICK

1. Choose two variables to represent the odd and even numbers to be selected.

2. Write algebraic expressions for the products formed by the first and second person and then add these two expressions.

3. If the number selected by the second person was even, will the sum in step #2 be even or odd?

4. If the number selected by the second person was odd, will the sum be even or odd?

5. Use your observations in steps #3 and #4 to explain the trick.

QUESTIONS TO PONDER

1. Could the multipliers 2 and 3 be replaced by 4 and 5 and still have the trick work?

2. Could you use any pair of even and odd numbers for multipliers?

3. Would the trick work if addition was replaced by subtraction? Addition by multiplication?

4. Could negative integers be used for the even and odd numbers selected by the subjects?

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FURTHER INVESTIGATIONS

1. Verify the trick using the "two case method" (i.e., in one case the first person chooses the odd number, in the next case, the even number).

2. The 1760 version of this trick refers to devising a "more secret" way for discovering whether the sum is even or odd. Devise such a way.

3. Will this trick work if the two persons are allowed to decide upon their own even and odd numbers?

4. Verify the following version of this trick.

   Someone holds two slips of paper, a slip with an odd number in one hand and a slip with an even number in the other. Have him multiply the number in the right hand by an odd number, and the number in the left hand by an even number. Have him tell you the sum of the two results. If the sum is even, the even number is in the right hand; if the sum is odd, the even number is in the left hand.

5. The UNVEILING EVEN AND ODD trick is often done with coins replacing the two given numbers. Devise a way to use two coins of different denominations for this trick.

6. Give several numerical examples to illustrate how the following variation of the trick works. Note that this variation is not based on whether the two numbers are even or odd.

   Suppose that the two numbers chosen are x and y. Also, suppose x is divisible by a known prime, p, and y has no common factors with the number x. After the two people, A and B, select the numbers x and y, have A multiply her/his number by a prime, q, which is different from the prime p. Then have B multiply by the prime p. Have them tell you the sum of these two products. If the sum is divisible by p, then A chose the number x. If the sum is not divisible by p, then B chose x.

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NUMBER 5 PREDICTING THE HOUR

The following trick appears in early mathematics and magic books. The trick can be used to predict the hour a person was born, the hour a person arises or goes to bed, or any hour suggested to a subject.

PERFORMING THE TRICK

1. Have a volunteer think of any number on the dial of a clock.
2. Instruct the volunteer to silently add one to the number s/he selected each time you point to a number on the clock dial until s/he gets to twenty.
3. Have the volunteer announce aloud when s/he reaches 20.
4. When 20 is announced, your pointer will be on the number selected by the volunteer.

HOW IT WORKS

When the subject is ready, you point to any seven numbers on the clock dial. Make sure your pointer lands on the 12 for your eighth number. Now, let your pointer follow the sequence 11, 10, 9, 8,... until the subject announces that 20 has been reached. When this happens, you will be pointing to the hour thought of by the subject.

AN EXAMPLE

Suppose the subject thinks of six o'clock. As you point to the sequence of numbers shown below, the subject counts silently from the selected number, 6.

Your Pointer	The Silent Count		Your Pointer	The Silent Count
3	7	Eighth Number ☛	12	14
9	8		11	15
4	9		10	16
8	10		9	17
11	11		8	18
4	12		7	19
2	13	Selected Number ☛	6	20

You are pointing to 6 when the subject announces the count of 20.

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WHY IT WORKS

Suppose the subject is thinking of the number x on the dial. After you point to seven numbers at random, the subject's count is up to $(x + 7)$.

Next you point to 12 and the subject's count becomes $(x + 8)$. Since the subject is instructed to count to 20, you will have $20 - (x + 8)$ or $(12 - x)$ more numbers to select before the subject announces that 20 is reached. You will now move your pointer consecutively backwards from 12, which is equivalent to subtracting the $(12 - x)$ from 12. After $(12 - x)$ times you will be pointing to $12 - (12 - x)$ or x, as the subject announces "20" aloud. This sequence of computations amounts to the equation $12 - (20 - (x + 8)) = x$, which does not depend upon the subject's choice of x.

Hints to Further Investigations

1. $12 - (15 - (x + 3)) = x$. You should be pointing to 12 for the third number.
2. For a 10-hour clock use the equation $10 - (20 - (x + 10)) = x$ if you wish the subject to silently count to 20. You should be pointing to 10 for the tenth number.

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THE TRICK: Predicting the Hour

1. Have your subject think of any number on the dial of a clock.
2. Tell the person to silently add one to the number s/he selected each time you point to a number on the clock dial, until s/he gets to 20. Have the person announce aloud when s/he reaches 20.
3. Point to 7 randomly selected numbers on the dial, then point to the 12 and continue pointing counterclockwise until 20 is announced.
4. When 20 is announced, you will be pointing to the number the person selected.

UNRAVELING THE TRICK

1. Choose a variable to represent the subject's selected hour.

2. Write algebraic expressions for the subject's silent count as you point to seven randomly selected hours on the clock face.

3. As you point to 12 as your next (eighth) number, write an expression for the subject's silent count.

4. Write an algebraic expression for the difference between the subject's current count and 20. Simplify this expression.

5. Now you begin moving your pointer consecutively counterclockwise through the counting numbers from 12. Use the algebraic expression in step #4 to explain why you will be pointing to the selected hour when the subject announces "20."

QUESTIONS TO PONDER

1. Are there any numerical restrictions on the chosen variable? Could the subject choose twelve o'clock?

2. If you started by pointing to seven o'clock and moved your pointer consecutively counterclockwise around the clock until the subject announced "20," would you be pointing to the subject's number?

3. If you were pointing to 12 for the sixth number and the subject counted silently to 18, would you predict the subject's number?

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FURTHER INVESTIGATIONS

1. To vary this trick you can change the count that the subject announces aloud. If you wish to have the subject count to 18 instead of 20, the trick-equation would become $12 - (18 - (x + 6)) = x$.

   Thus, you would make sure you are pointing to 12 on the sixth number. When the subject announces that 18 is reached, you would be pointing to x, the number selected by the subject.

   Write the trick-equation if the subject counts to 15. Explain the procedure for this variation of the game and verify the trick algebraically.

2. You may also vary the game for "clocks" with fewer than 12 hours or more than 12 hours. How would you vary the game for a "10-hour clock"?

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NUMBER 6 AMAZING PROGNOSTICATION

This trick seems to be in every magician's repertoire.

PERFORMING THE TRICK

1. Write a prediction (any number between 1 and 50) on a piece of paper and fold it.
2. Give the paper to someone to hold during the trick.
3. Ask a different person to choose any whole number between 50 and 100 and have her/him say it aloud.
4. Ask the person who chose the number to perform the calculations below.
   a. Add the two-digit number ___ to the number chosen.
   b. Delete the leftmost digit from the sum.
   c. Add the deleted digit to the remaining number.
   d. Subtract this result from the number originally chosen and announce the answer.
5. Ask the person holding the folded paper to read your prediction. The two numbers will match.

HOW IT WORKS

Once you have made your prediction, mentally subtract it from 99 and suggest that this number be added to your subject's number at the appropriate place in the trick. When all computations are completed, your prediction will match the subject's final answer.

AN EXAMPLE

Suppose your prediction is 43 and suppose your subject chooses 82. Since 43 from 99 is 56, you ask that 56 be added to the number selected by the subject.

Add 56 to the number chosen:

$$\begin{array}{r} 82 \\ +\ 56 \\ \hline 138 \end{array}$$

Delete the left digit and add it to the remaining number:

$$\begin{array}{r} \not{1}38 \\ +\ \ 1 \\ \hline 39 \end{array}$$

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Subtract this result from the number originally selected:

$$\begin{array}{r} 82 \\ -\ 39 \\ \hline 43 \end{array}$$

This matches your prediction.

WHY IT WORKS

Let your prediction be p, where $1 < p < 50$, and let n be your subject's choice, where $50 < n < 100$. You now suggest your subject add $99 - p$. Note that $49 < 99 - p < 98$.

The subject performs the following computations.

Add $99 - p$:	$n + (99 - p)$

Note that $99 < (n + 99 - p) < 198$ and therefore $(n + 99 - p)$ has a "1" in the hundred's column.

Delete the left digit:	$(n + 99 - p) - 100 = n - p - 1$
Add this digit:	$(n - p - 1) + 1 = n - p$
Subtract from the original number:	$n - (n - p) = n - n + p$
The result is your prediction.	$(n - n) + p = p$

Hints to Further Investigations

1. There are four cases to check. In one of these cases your prediction is 1 and the subject chooses 50. The computations below result from following the steps in the trick.

$$\begin{array}{rrrr} 99 & 50 & 48 & 50 \\ -\ 1 & +\ 98 & +\ 1 & -\ 49 \\ \hline 98 & 148 & 49 & 1 \end{array}$$

2. Subtract your prediction from 999 and follow the remaining steps in the original trick.

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THE TRICK: An Amazing Prognostication

1. Write a prediction (any number between 1 and 50) on a piece of paper and fold it.
2. Give the paper to someone to hold during the trick.
3. Ask a different person to choose any whole number between 50 and 100 and say it aloud.
4. Ask the person who chose the number to perform the calculations below.
   a. Add the two-digit number ___ to the number chosen.
   b. Delete the leftmost digit from the sum.
   c. Add the deleted digit to the remaining number.
   d. Subtract this result from the number originally chosen and announce the answer.
5. Ask the person holding the folded paper to read your prediction. The two numbers will match.

UNRAVELING THE TRICK

1. Choose one variable for your prediction and another for the subject's choice.
2. Subtract your prediction from 99 and add this difference to the subject's number (step #4 of the trick).
3. Use the algebraic expression in step #2 to verify that this represents a three-digit number with a 1 in the hundred's place.
4. Write an algebraic expression after the 1 in the hundred's place has been deleted (100 subtracted) and a 1 added to the remaining two-digit number. Simplify this expression.
5. Subtract this simplified expression from the subject's chosen number and use the result to explain the trick.

QUESTIONS TO PONDER

1. What are the numerical restrictions on the two selected variables?
2. Is the deleted digit always a 1?
3. Will the trick fail if the subject chooses a number less than 50?
4. Will the trick fail if your prediction number is greater than 50?

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FURTHER INVESTIGATIONS

1. Will this trick work if your prediction is either 1 or 50 and your subject chooses 50 or 100?

2. Alter this trick so that your prediction is between 100 and 200 and your subject can choose a number between 200 and 1000.

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NUMBER 7 PREDICTABLE DICE

I learned this trick from a magician. It requires a pair of six-sided dice, preferably with each die being a different color.

PERFORMING THE TRICK

1. Choose a volunteer and give her/him a pair of six-sided dice.
2. Designate one of the dice as DIE #1 and the other as DIE #2 (e.g., the white die is DIE #1, and the red die is DIE #2).
3. Instruct the volunteer to roll the dice, but not to show you what s/he rolls.
4. Have the volunteer find the following products:

   (top of DIE #1) × (top of DIE #2)

   (bottom of DIE #1) × (bottom of DIE #2)

   (top of DIE #1) × (bottom of DIE #2)

   (top of DIE #2) × (bottom of DIE #1).
5. Write the number 49 on a piece of paper and give it to a different person to hold. Announce: Without seeing what you rolled, I am writing the sum of these four products on this piece of paper.
6. Have the volunteer add the four products, then have her/him say the sum aloud.
7. Show the paper with your prediction.

HOW IT WORKS

No matter what two numbers are rolled, the sum of the four products will be 49.

AN EXAMPLE

Suppose a 6 is rolled on DIE #1 and a 3 on DIE #2. By examining a pair of dice you will see that a 1 appears on the bottom of DIE #1 and a 4 appears on the bottom of DIE #2.

The four products and their sum are:

$6 \times 3 = 18$

$1 \times 4 = 4$

$6 \times 4 = 24$

$1 \times 3 = 3$

The sum = 49

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WHY IT WORKS

First you should note that the number which appears on the top of a six-sided die added to the number on the bottom of that die is always 7. Thus, 1 is opposite 6, 2 is opposite 5, and 3 is opposite 4.

When the dice are rolled, let x = the number on top of DIE #1 and y = the number on top of DIE #2. From the observation above, it follows that the number on the bottom of DIE #1 is $(7 - x)$, since $7 + (7 - x) = 7$. Similarly, the number on the bottom of DIE #2 is $(7 - y)$.

The four products will be :

$$\begin{aligned} (x)(y) &= xy \\ (7-x)(7-y) &= 49 - 7x - 7y + xy \\ x(7-y) &= 7x - xy \\ y(7-x) &= 7y - xy. \end{aligned}$$

$$\begin{aligned} \text{The sum of these products} &= xy + 49 - 7x - 7y + xy + 7x - xy + 7y - xy \\ &= 49 \end{aligned}$$

The algebra shows that this sum does not depend on the values of x and y.

This outcome can also be verified using a proof by exhaustion, since there are a finite number of rolls for a pair of dice (thirty-six in all). One could verify that the sum of the four indicated products is 49 for each of the thirty-six possible rolls.

Hints to Further Investigations

1. Try several rolls (or simulated rolls) for a pair of dodecahedra dice. Inductively it will become clear that the sum of the four products is 169. If x represents the number on top of a dodecahedron, how would you express the number on the bottom? A verification of this trick can be made following the pattern of the algebraic proof used for standard dice. How many possible rolls are there for a pair of dodecahedra dice? Could you use a proof by exhaustion? Would you want to?
2. Again try several rolls (or simulated rolls) for icosahedra dice. What do you notice about the sum of the four products? Try the same technique of proof used previously. Could you use a proof by exhaustion?
3. The result is always 147 (3×49). This can be discovered inductively by trying several rolls of three dice. There are twelve products involved and the proof follows the same format as that used for two dice. How many possible rolls are there for three dice? How could you do a proof by exhaustion with a little help from your friends?

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THE TRICK: Predictable Dice

1. Roll a pair of six-sided dice.
2. Find the following products:

   (top of DIE #1) × (top of DIE #2)

   (bottom of DIE #1) × (bottom of DIE #2)

   (top of DIE #1) × (bottom of DIE #2)

   (top of DIE #2) × (bottom of DIE #1).
3. Find the sum of the products.
4. The sum of these four products can be predicted before the dice are rolled.

UNRAVELING THE TRICK

1. Assign a variable to the number showing on top of one die and a different variable to the number on top of the other die.

2. Examine a die. What is the sum of the two numbers on opposite sides of the die?

3. Using the variables chosen in step #1, write algebraic expressions for the numbers appearing on the bottom of each die.

4. Find the algebraic expressions for the four indicated products.

5. Find the sum of these four products. Explain how the trick works after algebraically simplifying the sum.

QUESTIONS TO PONDER

1. Are there any numerical restrictions on the two chosen variables?

2. If the sum of the numbers on the opposite sides of a die was not 7, would the trick work?

3. Does your proof work if a double is rolled?

4. How many possible outcomes are there when a pair of dice is rolled? Could you verify this trick using a proof by exhaustion?

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FURTHER INVESTIGATIONS

1. Polyhedra dice are commercially available. The placement of the numbers on these dice vary, however, depending on the manufacturer.

   On one pair of dodecahedra dice (12 pentagonal faces) the numbers 1 to 12 are paired as follows on each die.

TOP	BOTTOM	TOP	BOTTOM
1	12	4	9
2	11	5	8
3	10	6	7

   Use the fact that the sum of the top and bottom numbers is always 13 to alter the trick for a pair of dodecahedra dice.

2. One manufacturer numbers the 20 faces of their icosahedra dice (20 triangular faces) 0 to 9 twice. The numbers are paired on each die as follows.

TOP	BOTTOM	TOP	BOTTOM
0	9	5	4
1	8	6	3
2	7	7	2
3	6	8	1
4	5	9	0

   Use the fact that the sum of the top and bottom numbers is always 9 to alter the trick for a pair of icosahedra dice.

3. Suppose three standard six-sided dice are rolled and the sum of the following products of two numbers is found: (1) the products of all combinations of top numbers appearing on the three dice (i.e., top #1 × top #2; top #2 × top #3; top #1 × top #3), (2) the products of all combinations of bottom numbers appearing on the three dice, and (3) the products of the top number on each die with each of the two bottom numbers on the other dice. Can you predict the sum of these 12 products? Can you prove that your prediction always works?

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NUMBER 8 WHO IS HIDING THE COIN?

The following trick is based on a child's game called *Who Wears The Ring?* Arithmetic computation is used to discover the person in a group who is concealing an object. In the following version of the trick someone is concealing a coin.

PERFORMING THE TRICK

1. Choose a group of 2 to 9 people for the trick.
2. Assign an identification number (1, 2, 3, etc.) to each person in the group.
3. Turn your back to the group or leave the room for a moment.
4. Have the group choose one person to hide a coin in either her/his left or right hand. Each person in the group should know where the coin is.
5. Randomly choose one member of the group to mentally perform the following calculations:
   a. Multiply the identity number of the person concealing the coin by 2.
   b. Add 5 to this number.
   c. Multiply the result by 5.
   d. Add 10 to the result.
   e. Add 1 if the coin is in the person's right hand or 2 if the coin is in her/his left hand, and announce the result aloud.
6. From this result, you will not only identify who is hiding the coin but in which hand it is held.

HOW IT WORKS

Now, if everyone holds their closed fists in front of them, you can point to the hand concealing the coin. To do this, subtract 35 from the number announced to you. The ten's digit of the difference is the identity number of the person holding the coin and the unit's digit identifies the hand in which the coin is held (1 for right, 2 for left).

AN EXAMPLE

Suppose there are seven people (A, B, C, D, E, F, and G) and F hides the coin in his left hand. You assign numbers to the seven people: A is 1, B is 2, C is 3, etc.... Approach one person at random, say person B, and ask this person to follow the six steps outlined above. It might be necessary to review the identity numbers for the seven people. Person B's computations are shown on the next page.

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Step	Number Computed
1.	$2 \times$ (F's number) $= 2 \times 6 = 12$
2.	$12 + 5 = 17$
3.	$5 \times 17 = 85$
4.	$85 + 10 = 95$
5.	$95 + 2 = 97$
6.	Here B announces 97

After subtracting, $97 - 35 = 62$, you approach the person with identity number 6 (which is person F) and point to F's left hand.

WHY IT WORKS

Suppose the people are assigned the consecutive counting numbers (1, 2, 3, 4,...) as identification numbers and that the person with identity number x is hiding the coin in hand y, where $y = 1$ for the right hand and $y = 2$ for the left hand. Then:

Step	Number Computed
1.	$2x$
2.	$2x + 5$
3.	$5(2x + 5) = 10x + 25$
4.	$10x + 25 + 10 = 10x + 35$
5.	$10x + 35 +$ (1 or 2) depending on the hand holding the coin.

The number announced will be $10x + 36$ if the right hand is used or $10x + 37$ if the left hand is used. Now, you subtract 35 from the number announced and the answer will be $10x + 1$ or $10x + 2$. Note that the unit's digit identifies the hand (1 for right, 2 for left) and the ten's digit is the identity number of the person concealing the coin.

The end of this trick requires a slight modification if the number of people exceeds nine.

Hints to Further Investigations

1. If person #10, 11, or 12 is holding the coin, the final computation will produce a three-digit number. The first two digits will identify the person and the last digit the hand in which the coin is held.
2. Let x = the person's I.D. number, $y = 1$ or 2 depending on the hand used, and z = a digit from 1 to 6 identifying the type of coin. Following the steps in the trick, one arrives at the solution: $100x + 350 + 10y + z$. When you subtract 350 from this answer, you can identify x, y, and z.
3. Follow the algebraic analysis suggested in Trick #4, UNVEILING EVEN AND ODD.

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THE TRICK: Who is Hiding the Coin?

1. Select a group of 2–9 people and assign an identification number to each person.
2. When your back is turned, one person in the group hides a coin in her/his left or right hand. Each person in the group should know where the coin is.
3. Randomly choose someone in the group to perform the following calculations:
   a. Multiply the identity number of the person concealing the coin by 2.
   b. Add 5 to this number.
   c. Multiply the result by 5.
   d. Add 10 to the result.
   e. Now add 1 if the coin is in the person's right hand or 2 if the coin is in her/his left hand and announce the result.
4. From this result, you identify who is holding the coin and which hand it is in.

UNRAVELING THE TRICK

1. Assume each person is assigned a consecutive number for identification. Choose a variable to represent the I.D. number of the person holding the coin.

2. Write algebraic expressions for each successive step in the trick.

3. From the last algebraic expression subtract 35 and use the result to predict who is holding the coin and in which hand the coin is held.

QUESTIONS TO PONDER

1. Are there numerical restrictions on the variable chosen?

2. After 35 is subtracted from the reported number, will the result always be a two-digit number?

3. Could a number other than 10 be added in the trick's fourth computation? How would this affect the trick?

4. How would the outcome of the trick be affected if you added 1 for the right hand and did no further computation if the left hand was used?

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FURTHER INVESTIGATIONS

1. Try the trick for 12 people and explain the "slight modification" needed to predict the person concealing the coin.

2. To complicate the trick, allow the person to conceal a penny, nickel, dime, quarter, half-dollar, or silver dollar. Substitute the following steps after step #3e in the trick's instructions on the previous page.

   f. Multiply the result by 10.

   g. Add the appropriate one-digit number:
      1 for a penny
      2 for a nickel
      3 for a dime
      4 for a quarter
      5 for a half-dollar
      6 for a dollar

   h. Announce the result.

   In our example, suppose a quarter was concealed in the left hand of person #6. Then step (f) results in $10 \times 97 = 970$ and step (g) gives $970 + 4 = 974$. When this result is announced, you subtract 350 and the answer 624 identifies "person #6," "the left hand," and "a quarter is concealed." Use algebra to show why this variation of the trick works.

3. Use algebra to explain how the following version of this trick works. (Hint: Refer to Trick Number 4, UNVEILING EVEN AND ODD).

   Ask someone to put a dime in one hand and a penny in the other.

   Ask her/him to multiply the value of the coin in her/his right hand by 4, 6, or 8, and the value of the coin in the left hand by 3, 5, or 7.

   Now ask her/him to add the results and tell you the total. If the total is even, s/he has the penny in the right hand; if the total is odd, s/he has it in the left hand.

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NUMBER 9 THE MARBLE TRANSFER

This trick is adapted from an early arithmetic recreation. Here it is presented with two subjects, a male (A) and a female (B), and a bowl of marbles. In practice a container of any small objects may be used. (You need at least 80 marbles to insure that this trick will work.)

PERFORMING THE TRICK

1. Choose a male (A) and female (B) volunteer. Turn your back to the volunteers and the bowl of marbles and give them instructions a–d below.
   a. A, take a handful of marbles from the bowl and count them but do not tell me how many you have.
   b. B, take from the bowl 3 times as many marbles as A took.
   c. A, give 5 of your marbles to B.
   d. B, give back to A 3 times as many marbles as A is now holding.
2. At this point, announce the exact number of marbles that B is holding.
3. Ask A how many marbles he has.
4. Announce the number of marbles A and B originally took from the bowl.

HOW IT WORKS

The first announcement (marbles B is holding) is 20 marbles. It is always 20 in this trick. When A tells you how many he has, add 20 to this number and divide the sum by 4 to predict the original number of marbles taken by A. B originally took 3 times this number.

AN EXAMPLE

Suppose A takes 15 marbles. Then B takes 3×15 or 45 marbles.

A gives 5 marbles to B, leaving A with 10 and B holding 50. Now B gives $3 \times 10 = 30$ marbles to A, leaving B with 20 and A holding 40. At this point announce that B has 20 and ask A how many he has. When A says 40, compute $(40 + 20) \div 4 = 15$ and announce that A originally took 15 marbles and B took 45.

WHY IT WORKS

Let n represent the number of marbles taken by subject A.

Subject B takes $3n$ marbles. After the first transfer of 5 marbles, A has $(n - 5)$ marbles and B has $(3n + 5)$ marbles. The second transfer involves $3(n - 5)$ marbles from B to A. After this transfer, A has $(n - 5) + 3(n - 5) = 4n - 20$ marbles and B has $(3n + 5) - 3(n - 5) = 5 + 15 = 20$ marbles. (Steps summarized in diagram next page.)

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Here you are able to announce that B holds 20 marbles. Suppose x is the number A is now holding.

When A tells you the value of x, solve the equation: $x = 4n - 20$ for n, the number of marbles originally taken from the bowl. This involves adding 20 to x and dividing this sum by 4. The answer to this is the number of marbles originally taken from the bowl by A. Whatever this number is, you need only multiply it by 3 (as was done in the trick's directions) to tell B how many she originally took from the bowl.

Number of Marbles A and B Have at Each Step		
	A	**B**
Original Draw	n	3n
First Transfer	n – 5	3n + 5
Second Transfer	(n – 5) + 3(n – 5) = 4n – 20	(3n + 5) – 3(n – 5) = 20

Hints to Further Investigations

1. Assume there are a lot of marbles in the bowl. Would the trick work if A originally took 2 marbles? 3? 4? 5? More than 5? To ensure that the trick will work, restate the first direction to A so he will have enough marbles to make the trick possible.
2. Since you know B has 20 marbles, any transfers or computations that follow will allow you to keep track of the number held by B.
3. Simulate the trick for several different selections by subject A. Can you use the same method of proof used in the original trick?
4. Suppose A takes n marbles. Then B takes pn marbles. After the first transfer, the number held by A and B, respectively, are $n - q$ and $pn + q$. After the second transfer, the number held by each is $(n - q) + p(n - q)$ and $pn + q - [p(n - q)]$. Show this last expression is equal to $q(1 + p)$. When A announces that he holds x marbles, show that the number of B's marbles added to x and then divided by $(1 + p)$ is the original number taken by A.
5. First transfer 10 white marbles from Bowl #1 to Bowl #2. Now let x be the number of black marbles and y the number of white marbles involved in the transfer from Bowl #2 to Bowl #1. How are x and y related? Write algebraic expressions for the number of white marbles in each bowl. Do the same for black. Compare these expressions.

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THE TRICK: The Marble Transfer

1. Person A takes a handful of marbles from a bowl and counts them.
2. Person B takes 3 times as many marbles as A took.
3. Person A gives 5 marbles to person B.
4. Person B gives back to person A 3 times as many marbles as A is now holding.
5. At this point, you can predict the number of marbles Person B is holding.
6. When you know how many marbles person A has now, you can determine how many marbles both A and B originally took.

UNRAVELING THE TRICK

1. Choose a variable to represent the number of marbles taken by subject A and write an expression for the number taken by B.

2. After each marble transfer, write algebraic expressions for the number of marbles held by subject A and by subject B.

3. Simplify the last expressions to explain why subject B holds 20 marbles at the end of the trick.

4. Show that you can predict the original number taken by A if you add 20 to the final number held by A and divide this sum by 4.

QUESTIONS TO PONDER

1. Does the trick depend upon the original number of marbles in the bowl? Is there a minimum number necessary to perform the trick?

2. Are there any numerical restrictions on the number of marbles chosen by A?

3. If there were 100 marbles in the bowl to start, what is the maximum number A could select? The minimum number?

4. How could the directions be made more specific so that the trick will always work?

5. If there were a fixed number of marbles in the bowl (say 100), could you verify the trick using a proof by exhaustion?

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FURTHER INVESTIGATIONS

1. For what values of n (the number of marbles originally taken by subject A) will this trick fail? How can you assure yourself that this will not happen?

2. After the two transfers of marbles take place, you know B is holding 20 marbles. Think of another way to end this trick and explain your prediction or predictions.

3. How would your predictions be affected if (1) B took 4 (instead of 3) times as many marbles as A, and (2) the first transfer of marbles was 7 marbles (instead of 5), and (3) the second transfer was 4 (instead of 3) times the number now held by A?

4. In the original trick, use p as the multiplier (in place of 3). Also, use q as the first transfer value (in place of 5). Verify with algebra that the first prediction should be $q(1 + p)$. Also, verify that the original number of marbles taken by A will be $(x + q(1 + p)) \div (1 + p)$, where x is the number of marbles last held by A.

5. Here is another marble transfer trick. Bowl #1 contains 20 white marbles and Bowl #2 contains 20 black marbles. Ten marbles are randomly selected from Bowl #1 and transferred to Bowl #2. Then ten marbles are randomly selected from Bowl #2 and transferred to Bowl #1. Use algebra to show that there are as many white marbles in Bowl #2 as there are black marbles in Bowl #1.

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NUMBER 10 THE MONEY MYSTERY

The following trick is often referred to as the "1089" trick for reasons you will discover. Since it is presented here using a sum of money, it becomes The Money Mystery.

PERFORMING THE TRICK

1. Write $10.89 on a piece of paper and seal it in an envelope. Choose a volunteer and give her/him directions a–e below.
   a. Imagine a sum of money less than ten dollars ($10.00).
   b. Transpose the digits.
   c. Take the difference between these two amounts.
   d. Transpose the digits in the difference.
   e. Add this transposition to the difference.
2. Reveal the sealed answer.

HOW IT WORKS

The answer is always $10.89. Thus, before the amount is selected, the answer ($10.89) can be written on paper and sealed in an envelope. When the computations are completed, the envelope is opened and the solution revealed.

AN EXAMPLE

If the amount chosen is $6.73, the transposition of the digits is $3.76. The difference is then computed.

$$\begin{array}{r} 6.73 \\ -\ 3.76 \\ \hline 2.97 \end{array}$$

Transpose 2.97 (7.92) and add it to the answer.

$$\begin{array}{r} 2.97 \\ +\ 7.92 \\ \hline 10.89 \end{array}$$

WHY IT WORKS

The explanation of this trick requires that the three digits in the chosen number be listed by place value. This is shown in the table on the next page.

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		Place Value		
	Step	Units	Tenths	Hundredths
1	Choose your number.	x	y	z
2	Borrow 1 from the units column.	x – 1	y + 10	z
3	Borrow 1 from the tenths column.	x – 1	y + 9	z + 10
4	Reverse the digits in step 1.	z	y	x
5	Find the difference of steps 3 and 4 and simplify.	x – 1 – z x – 1 – z	y + 9 – y 9	z + 10 – x z + 10 – x
NOTE: In step 5, if x = z, then the difference is -1 + .9 + .10 = 0				
		Expanded Notation		
6	Write step 5 in expanded notation.	1(x – 1 – z) + 1/10(9) + 1/100(z + 10 – x)		
7	Reverse the digits in step 6.	1(z + 10 – x) + 1/10(9) + 1/100(x – 1 – z)		
8	Find the sum of steps 6 and 7.	1(x – 1 – z + z + 10 – x) + 1/10(9 + 9) + 1/100(z + 10 – x + x – 1 – z)		
9	Simplify step 8.	1(9) + 1/10(18) + 1/100(9)		
10	Write step 9 in decimal form.	9 + 1.8 + .09 = $10.89		

As noted in step 5, this works as long as the first and last digits are not equal. In that case the subtraction gives a difference of zero.

Hints to Further Investigations

1. Make sure you transpose all three digits. Thus, $0.01 transposed becomes $1.00.
2. If you choose 364, the transposition is 463 and the difference, 99, is not a three-digit number. Does the trick fail here? To avoid this situation, you might simply suggest that the hundred's and unit's digits differ by more than 1.
3. Try the trick in each of these cases, but make sure the difference between the leftmost and rightmost digits is greater than 1. Do you see a pattern to the answers?
4. If you want the final result to be 1991, what number would you add, and at what step in the trick would you add it?
5. Must the difference between the yard and inch measures be more than 1? Try 3 yards, 5 feet, and 4 inches? Could the yard and inch measures be the same?

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THE TRICK: The Money Mystery

1. Imagine a sum of money less than ten dollars ($10.00).
2. Transpose the digits.
3. Take the difference between these two amounts.
4. Transpose the digits in the difference.
5. Add this transposition to the difference.
6. You can predict the final answer.

UNRAVELING THE TRICK

1. Construct a three-column (place value) table with labels: "units," "tenths," and "hundredths." Place variables in the table to represent the three digits in the chosen amount of money.
2. Borrow 1 from the unit's column and rewrite the expressions in the three columns.
3. Similarly, borrow 1 from the tenths' column.
4. Reverse the three chosen digits (variables).
5. Find the difference between the expressions in steps #3 and #4 and simplify the results.
6. Use expanded notation to express the number represented by the three digits in step #5.
7. Rewrite the number in step #6 with the digits reversed.
8. Add the two expressions (step #6 and #7), simplify the answer, and write it in standard decimal form. Does this result explain the trick?

QUESTIONS TO PONDER

1. What numerical restrictions are necessary on the digits chosen in the original amount of money? Could you use $3.53? Why or why not?

2. Using the algebraic expressions in step #5 above (*Unraveling the Trick*), can you explain the numerical restrictions on the variables (digits)?

3. What is the smallest amount of money that could be chosen to start the trick? The largest amount? Why?

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FURTHER INVESTIGATIONS

1. Even if you choose one cent ($0.01) as the starting amount, the computations lead to $10.89. Show that this is true.

2. A magician friend suggested asking someone to write a three-digit number and follow the steps outlined for THE MONEY MYSTERY trick. For example, a choice of 659 leads to the following computations.

$$\begin{array}{r} 956 \\ -\ 659 \\ \hline 297 \\ +\ 792 \\ \hline 1089 \end{array}$$

   This can be a little dangerous, however. Show that the trick using a three-digit number fails when the first and last digit of the number are the same or differ by only one. Reword the problem so that a three-digit number could be used.

3. What would your prediction be if a four-digit number was chosen? A five-digit number? A six-digit number? etc.

4. To disguise the trick (so the sum will not always be 1089) alter the steps so an additional number (of your choice) is added to the figures and show how you can then predict the answer to the trick.

5. Try varying the game by writing a measurement in yards, feet, and inches and predicting the answer. For example, if you chose 10 yards, 2 feet, and 8 inches the computations would appear as below.

Yards	Feet	Inches
10	2	8
− 8	− 2	− 10

   To subtract you must borrow from the feet and yards columns as show below.

Yards	Feet	Inches
9	4	20
− 8	− 2	− 10
1	2	10
+ 10	+ 2	+ 1
11	4	11

   = 12 yards, 1 foot, and 11 inches.

   Will the answer always be 12 yards, 1 foot, and 11 inches? What limitations would have to be placed on the chosen measurements to insure that the trick will work?

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NUMBER 11 PERMUTATION PREDICTION

The following is a trick well known to magicians. It has appeared in both mathematics journals and texts since the nineteenth century. Some variations of the trick are presented in *Further Investigations*.

PERFORMING THE TRICK

1. Choose a volunteer and give her/him instructions a–c below.
   a. Write down a number consisting of three consecutive ascending digits but don't tell me the number.
   b. Figure out all the possible arrangements of these three digits and write them down. There will be six of them including your original number.
   c. Add these numbers and divide the sum by 6.
2. Have the volunteer reveal the quotient.
3. From this quotient, predict the number originally selected.

HOW IT WORKS

No matter what number a person originally chooses in this trick, all three digits of the final quotient will be identical. The original number has this repeated digit in the ten's place. The other two digits will be one more and one less than this digit.

AN EXAMPLE

If the number selected is 345, the permutations are: 354, 435, 453, 534, and 543. After completing the computations below, the result 444 is reported.

$$\begin{array}{r} 345 \\ 354 \\ 435 \\ 453 \\ 534 \\ +\ 543 \\ \hline 2664 \end{array} \qquad \begin{array}{r} 444 \\ \hline 6\overline{)2664} \end{array}$$

At this point you compute the first and third digits, (4 – 1) and (4 + 1), and announce that the original number was 345.

WHY IT WORKS

There are two ways to explain why this trick always works.

1. Since the three digits of the chosen number must be consecutive, represent them by

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a, $a + 1$, and $a + 2$. Note that $1 \leq a \leq 7$ (because 789 is the largest number you can use). The six permutations are listed in the table below.

	Hundred's place	Ten's place	Unit's place
The six permutations	a	a + 1	a + 2
	a	a + 2	a + 1
	a + 1	a	a + 2
	a + 1	a + 2	a
	a + 2	a	a + 1
	a + 2	a + 1	a
Column sums	6a + 6	6a + 6	6a + 6

The sum of the six permutations in expanded notation is

$$S = 100(6a + 6) + 10(6a + 6) + (6a + 6) = 600(a + 1) + 60(a + 1) + 6(a + 1).$$

Now, dividing the sum, S, by 6 you get $S/6 = 100(a + 1) + 10(a + 1) + (a + 1)$. This means that the answer will have $(a + 1)$ as a repeated digit. From this you can predict the three digits of the original number: a, $a + 1$, and $a + 2$.

2. Another way to explain how this works is using proof by exhaustion. Note that there are only 7 possible starting numbers: 123, 234, 345, 456, 567, 678, and 789. You can verify by computation that each of these 7 possible starting numbers will yield an answer with 3 repeated digits. Also you can verify that the repeated digit is always the ten's digit of the original number.

Hints to Further Investigations

1. Try the same numerical and algebraic analysis used in the original trick. You'll need to divide by 12.
2. The result is nice because it yields a single-digit predictor.
3. The answer is always a constant.
4. There are 24 permutations of a four-digit number. For each of the 6 possible starting numbers, consider the results when the sum of the permutations is divided by 12. Combine these results with an algebraic analysis. (Column sums divided by 12 are all $2a + 3$.)

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THE TRICK: Permutation Prediction

1. Choose any number consisting of three consecutive ascending digits.
2. Determine all permutations (possible arrangements) of these digits and list these six three-digit numbers.
3. Compute the sum of these numbers and divide the sum by 6.
4. From the quotient, the original number can be predicted.

UNRAVELING THE TRICK

1. Using a single variable, write an algebraic expression for each digit in the original number. Write the expressions across the page.

2. Write the six possible arrangements for each of the three expressions you just found. Line up the expressions for the six permutations in three columns.

3. Find the sum of each column and divide each sum by 6.

4. Use the results of the last step to explain the trick.

QUESTIONS TO PONDER

1. What numerical restrictions are there on the variable you chose?

2. Why do we divide by 6? Would another number work as well?

3. What is the smallest and largest number that we can use for the last digit in the chosen number?

4. What is the smallest and largest number that we can use for the first digit in the chosen number?

5. Could we use $a - 1$, a, and $a + 1$ to represent the digits in the number? a, $a + 1$, and $a + 2$? $a + 2$, $a + 3$, and $a + 4$?

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FURTHER INVESTIGATIONS

1. What would happen if you choose 3 consecutive even (or odd) digits? Why does or doesn't this work?

2. Here is a variation on ending the trick. Consider adding the following two computations after completing the division by 6. Divide the quotient by 3 and then divide the new quotient by 37. Experiment to see the effect of these computations on the trick. How does each step affect the result? Can you predict the original number from the final answer? Why or why not?

3. Here is another possible way to end the trick. After the original two computations, ask the subject to divide the quotient by the sum of the three original digits. Is this result predictable? Why or why not?

4. Examine what happens when a three-digit number is replaced by a four-digit number. In this case can you find a pattern in the answers which allows you to predict the original number?

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NUMBER 12 DIVINING TWO NUMBERS

Every magician knows a trick to predict (or divine) more than one number thought of by a subject. Here is an example of a trick to divine two numbers. Because the trick uses a pair of dice to generate the two numbers "thought of," the numbers are naturally restricted to single digits from 1 to 6.

PERFORMING THE TRICK

1. Have a volunteer roll a pair of dice. Then give her/him instructions a–e below.
   a. Multiply the number on top of one die by 2.
   b. Add 3 to this.
   c. Multiply by 5.
   d. To this result add the number on top of the second die.
   e. Multiply this sum by 10.
2. Have the subject announce her/his answer and from that you will reveal the two numbers rolled.

HOW IT WORKS

When the subject reveals the answer, mentally subtract 150 from the answer. The digits in the hundred's and ten's places of the difference are the numbers rolled on the dice. A zero will always be in the unit's digit.

AN EXAMPLE

If a 5 and a 3 were rolled, these computations follow:

$$2 \times 5 = 10$$
$$10 + 3 = 13$$
$$5 \times 13 = 65$$
$$65 + 3 = 68$$
$$68 \times 10 = 680.$$

When 150 is subtracted from 680, the result (530) reveals that the subject rolled a 5 and a 3.

A variation of this trick replaces the dice with a domino chosen at random from a set of dominoes. The subject is directed to hold the domino vertically so that one number is

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above the other and to follow the directions listed in the trick above using the top number as the first number in the trick. The prediction can be more exact in this case. After you subtract 150, the hundred's digit is the top number and the ten's digit is the bottom number on the domino.

WHY IT WORKS

Suppose the numbers on the dice (or domino) are represented by x and y. Then these computations follow.

Multiply one number (top number) by 2:	$2x$
Add 3 to this:	$2x + 3$
Multiply by 5:	$5(2x + 3) = 10x + 15$
Add the second number:	$10x + 15 + y$
Multiply this sum by 10:	$10(10x + 15 + y) = 100x + 10y + 150$.

When the last number is reported, subtract 150, and the result is always $100x + 10y + 0$. Thus, x and y are the digits in the hundred's and ten's places, allowing you to announce them to your subject. A zero will always be in the unit's place.

Hints to Further Investigations

1. To have the three numbers rolled on the dice appear in the answer as separate digits, somewhere in the trick you must multiply by 10 and also by 100. Use the algebraic analysis of the original trick as a model.
2. If all the numbers appearing on the dice are less than 10, will the trick work as presented? If any of the numbers are greater than 10, how could you alter the trick?
3. If x, y, and z represent the three chosen numbers, the algebraic result of the directions given in the trick is $100x + 10y + z + 55$.

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THE TRICK: Divining Two Numbers

1. Have a volunteer roll a pair of dice. Then give her/him instructions a–e below.
   a. Multiply the number on top of one die by 2.
   b. Add 3 to this.
   c. Multiply by 5.
   d. To this result add the number on top of the second die.
   e. Multiply this sum by 10.
2. From the answer, you can tell the two numbers rolled.

UNRAVELING THE TRICK

1. Use two different variables to represent the numbers rolled on the dice.

2. Write algebraic expressions to represent the resulting numbers after each computational step in the trick.

3. Subtract 150 from the last algebraic expression in step #2 and use this result to explain the trick.

QUESTIONS TO PONDER

1. What numerical restrictions are there on the variables you choose?

2. If instead of rolling a pair of dice the subject chose two numbers, what restrictions would have to be placed on the choices?

3. What is the largest number for which the trick (as presented) will work? The smallest number?

4. Could you use dodecahedra (12-sided) or icosahedra (20-sided) dice for the trick?

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FURTHER INVESTIGATIONS

1. Develop a similar trick for rolling three dice.

2. Develop a similar trick for rolling a pair of icosahedra (20-sided) or other type dice.

3. Analyze (algebraically) why the following divining trick works.

   Think of three numbers each of which is less than 10.

   Suppose the numbers thought of are 2, 3, and 5.

   Double the first number ..... $2 \times 2 = 4$

   Add 1 to it ..... $4 + 1 = 5$

   Multiply by 5 ..... $5 \times 5 = 25$

   To this add the second number ..... $25 + 3 = 28$

   Double this number ..... $28 \times 2 = 56$

   Add 1 to it ..... $56 + 1 = 57$

   Multiply by 5 ..... $57 \times 5 = 285$

   To this add the third number ..... $285 + 5 = 290$

   Report the resulting number 290

   From the number reported subtract 55 ..... $290 - 55 = 235$

   The number remaining (235) is the three digits thought of—2, 3, and 5.

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NUMBER 13 REVEALING FACTORS

Students are aware that a number is divisible by 2 if it ends in an even digit. This rule is one of many divisibility rules found in textbooks. The following simple trick uses divisibility rules for 2, 3, 4, 5, 6, 8, 9, 10, 11, and 12. Note that 7 is omitted. Divisibility rules for 7 are too complex to apply quickly.

PERFORMING THE TRICK

1. Have someone write down a whole number.
2. Immediately produce a list of numbers which evenly divide the given number.

HOW IT WORKS

Below is a list of divisibility rules for a given number, *C*.

C is divisible by:

2 if 2 divides the unit's digit of *C*

3 if 3 divides the sum of the digits of *C*

4 if 4 divides the two-digit number represented by the last two digits of *C*

5 if *C* ends in 0 or 5

6 if *C* is divisible by both 2 and 3

8 if 8 divides the number represented by the last three digits of *C*

9 if 9 divides the sum of the digits of *C*

10 if *C* ends in 0

11 if 11 divides (the sum of the odd place digits) – (the sum of the even place digits)

12 if *C* is divisible by both 3 and 4

AN EXAMPLE

Given 384, the following list of divisors can be immediately produced using the divisibility properties above.

2 since 2 divides 4

3 since 3 divides 3 + 8 + 4 or 15

4 since 4 divides 84

6 since both 2 and 3 divide 384

8 since 384 ÷ 8 = 48

12 since both 3 and 4 divide 384

56

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Given 290,136 the following list of divisors can be produced:

2 since 2 divides 6

3 since 3 divides 2 + 9 + 0 + 1 + 3 + 6 or 21

4 since 4 divides 36

6 since both 2 and 3 divide 290,136

8 since 8 divides 136

11 since 11 divides (9 + 1 + 6) – (2 + 0 + 3) or 11

12 since both 3 and 4 divide 290,136

WHY IT WORKS

Two of these rules, divisibility by 8 and divisibility by 9 will be verified using algebra. Other divisibility rules are presented as investigations for the reader. Verification of all these divisibility properties depends on the following properties of whole numbers.

> *If a number divides each of two given numbers, it will divide the sum of the numbers and the difference of these two numbers. For example, 3 divides both 36 and 9, therefore 3 will divide (36 + 9) = 45 and 3 will divide (36 – 9) = 27.*

The verification of divisibility by 8 starts with a five-digit number, but the argument easily generalizes to any number of digits. Suppose we have a five-digit number, $C = abcde$. In expanded notation we write $C = 10000a + 1000b + 100c + 10d + e$. The first two terms of C can be factored as follows: $10000a + 1000b = 1000(10a + b)$. Since 8 divides 1000, 8 will divide $1000(10a + b)$ or the first two terms of C. Now, if we know that 8 divides the last three digits of C, $(100c + 10d + e)$, then 8 will divide the sum of the first two and the last three terms, that is, 8 will divide C.

To verify the divisibility rule for nine, again assume $C = abcde$ which is $10000a + 1000b + 100c + 10d + e$ in expanded notation. Now consider the following divisions of the powers of 10 by nine and the related equations, which are normally used to check division computations. (r = remainder)

$$1 \div 9 = 0\ r\ 1 \ldots\ldots 1 = 9 \times 0 + 1$$
$$10 \div 9 = 1\ r\ 1 \ldots\ldots 10 = 9 \times 1 + 1$$
$$100 \div 9 = 11\ r\ 1 \ldots\ldots 100 = 9 \times 11 + 1$$
$$1000 \div 9 = 111\ r\ 1 \ldots\ldots 1000 = 9 \times 111 + 1$$
$$10000 \div 9 = 1111\ r\ 1 \ldots\ldots 10000 = 9 \times 1111 + 1$$

Multiplying the second column of equations by e, d, c, b, and a, respectively, we have:

$$1e = 9 \times 0e + 1e$$
$$10d = 9 \times 1d + 1d$$
$$100c = 9 \times 11c + 1c$$
$$1000b = 9 \times 111b + 1b$$
$$10000a = 9 \times 1111a + 1a$$

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Adding these five equations we have: $(10000a + 1000b + 100c + 10d + 1e) = (9 \times 0e + 9 \times 1d + 9 \times 11c + 9 \times 111b + 9 \times 1111a) + (e + d + c + b + a)$.

The expression on the left side of this equation is C. On the right side we can factor a 9 from the first set of parentheses, giving:

$$C = 9(0e + 1d + 11c + 111b + 1111a) + (e + d + c + b + a).$$

We know 9 divides the first term of this expression. If 9 divides the sum of the digits of C (the second term), then 9 must divide the sum of these two terms, which is C.

Hints to Further Investigations

1. The divisors up to 12 are as follows.

   Divisors of 3960: 2, 3, 4, 5, 6, 8, 9, 10, 11, 12

   Divisors of 19384: 2, 4, 8, and the only other divisor is 2423

   Divisors of 123321: 3, 11, and the primes 37 and 101

   Divisors of 2310: 2, 3, 5, 6, 10, 11, and 7

2. $C = 10000a + 1000b + 100c + 10d + e = 4[2500a + 250b + 25c] + [10d + e]$. If 4 divides $[10d + e]$, it will divide the right-hand side of the equation which means 4 will divide C.

3. Start by writing:

$$1 \div 3 = 0\ r\ 1 \ldots . 1 = 3 \times 0 + 1$$
$$10 \div 3 = 3\ r\ 1 \ldots . 10 = 3 \times 3 + 1$$
$$100 \div 3 = 33\ r\ 1 \ldots . 100 = 3 \times 33 + 1$$
$$1000 \div 3 = 333\ r\ 1 \ldots . 1000 = 3 \times 333 + 1$$
$$10000 \div 3 = 3333\ r\ 1 \ldots . 10000 = 3 \times 3333 + 1$$

   Use the set of equations on the right and follow the outline for the derivation of the divisibility property of 9.

4. Multiply the five given equations by e, d, c, b, and a, respectively. Add the resulting equations and simplify as follows: $10000a + 1000b + 100c + 10b + a = 11[0 \times e + 1 \times d + 9 \times c + 91 \times b + 909 \times a] + [e - d + c - b + a]$.

   Now, if 11 divides $[e - d + c - b + a]$, then 11 divides the right-hand side of the equation which means it divides the number.

5. The first and third computations are correct.

6. For a simple example, reverse the first two digits in the answer to the third computation in #5 above: $668 \times 407 = 721876$.

7. Casting-out nines in division may appear as below—$2 \times 2 = 4$ checks the division.

$$\begin{array}{rcl} & 407 & 2 \\ 2 & 668\overline{)271876} & 4 \end{array}$$

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THE TRICK: Revealing Factors

1. Have someone write down any whole number.
2. Using the rules of divisibility, immediately produce a list of numbers each of which evenly divides the given number.

UNRAVELING THE TRICK

(Divisibility of *abcde* by 8)

1. Write the five-digit number, *abcde*, in expanded notation.
2. Rewrite the expression in step #1 by factoring an 8 out of the first two terms.
3. Assume 8 divides [$100c + 10d + e$]. Use this fact and steps #1 and #2 above to verify the rule for divisibility of a number by 8.

(Divisibility of *abcde* by 9)

1. If 1 is divided by 9, the result may be written: $1 \div 9 = 0\ r\ 1$ which yields the equation $1 = 9 \times 0 + 1$. Write similar equations obtained by dividing 10, 100, 1000, and 10000 by 9.
2. Multiply the five resulting equations from step #1 by *e*, *d*, *c*, *b*, and *a*, respectively.
3. Add these five equations.
4. On the right-hand side of this sum, factor a 9 from the first five terms.
5. Assume 9 divides the sum of the digits, [$a + b + c + d + e$]. Use this fact and the result of step #4 to verify the divisibility rule for 9.

QUESTIONS TO PONDER

1. Could the arguments for divisibility by 8 and by 9 be generalized to numbers with more than five digits?
2. Could the argument for divisibility by 8 be modified to establish a divisibility rule for 4?
3. Could the argument for divisibility by 9 be modified to establish a divisibility rule for 3?

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FURTHER INVESTIGATIONS

1. Based on the divisibility rules, write a list of divisors for each of the following numbers: 3960, 19384, 123321, and 2310.

2. Verify the divisibility rule for 4 (apply the technique used for "8" on the previous page).

3. Verify divisibility by 3 (apply the technique used for "9" on the previous page).

4. Verify the divisibility rule for 11 applying the technique of "9" on the following set of equations.

$$\begin{aligned} 1 &= 11 \times 0 + 1 \\ 10 &= 11 \times 1 - 1 \\ 100 &= 11 \times 9 + 1 \\ 1000 &= 11 \times 91 - 1 \\ 10000 &= 11 \times 909 + 1 \end{aligned}$$

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5. Early arithmetic texts used "casting out nines" as a method of checking arithmetic computations. To cast out nines in a number, add the digits of the number, and keep adding the digits in the sum until you reach a one-digit number. If this digit is nine, "cast it out" by making it a zero, otherwise, retain the single digit number.

   Casting out nines in a given number:

   3847 .....3 + 8 + 4 + 7 = 22.....2 + 2 = 4

   135 .....1 + 3 + 5 = 9 0

   26 .....2 + 6 = 8 8

   We will now use casting out nines to check the following multiplication problem: $34 \times 26 = 884$.

   To check the multiplication, cast out nines in both factors (34 and 26). Multiply the two resulting numbers and cast out nines from the product. Compare this result with the product, 884, after casting out nines (as shown below).

   34 ....3 + 4 = 7

   × 26 ....2 + 6 = 8 $7 \times 8 = 56$ ...5 + 6 = 11 ....1 + 1 = 2

   884 ....8 + 8 + 4 = 20 .....2 + 0 = 2

   Check the following computations by casting out nines. (Which computations are correct and which incorrect?)

387	1085	668
× 143	× 987	× 407
55341	1069895	271876

6. Present a numerical example to illustrate how casting out nines can go wrong.

7. Present numerical examples to illustrate how casting out nines can be used to check division, addition, and subtraction.

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NUMBER 14 MAGICAL NUMBER NINE

Casting out nines uses a familiar property of the number nine to check arithmetic computations (see Trick Number 13). Properties of nine also are the basis for numerous number tricks. Here are variations on such a trick.

PERFORMING THE TRICK

Choose a volunteer and give her/him the directions under Variation One or Two below.

Variation One

1. Select a three-digit number in which none of the three digits is the same.
2. Form a different three-digit number by rearranging the original digits.
3. Now find the difference of the two three-digit numbers.
4. Add the digits in this difference.
5. Tell me the first (or last) digit in your sum and I will tell you the other digit.

Variation Two

1. Follow the steps above up to step 4.
2. If the sum is not one digit, continue adding the digits in the result until you reach a one-digit number. I will predict this number.

HOW IT WORKS

Variation One

When you subtract the two three-digit numbers, the difference will always be a multiple of 9. When you add the digits, the result will also be a multiple of nine. Since you know the first digit and know that the number is a multiple of 9, you can predict the other digit.

Variation Two

The digit will always be 9.

AN EXAMPLE

Variation One

If the number selected is 357 and 735 is chosen as the permutation, these computations follow: 735 – 357 = 378 and 3 + 7 + 8 = 18. Since the result is a multiple of 9, when you are told the first digit, 1, you know that the last digit is 8.

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Variation Two

Suppose 912 is chosen and the permutation selected is 129. Then: 912 – 129 = 783 and 7 + 8 + 3 = 18 and 1 + 8 = 9. In this variation, the result will always be 9.

WHY IT WORKS

Because the trick allows the subject to choose any permutation of the original three digits, the verification in algebra requires examining several possible cases. I will illustrate one such case and you may wish to try others.

Suppose the three-digit number chosen is written *abc*, where $a \neq b$, $b \neq c$, and $a \neq c$. Let us assume that $a > c$. Also, let us assume that the permutation of the digits considered is *cba*. To subtract *cba* from *abc* requires borrowing from the hundred's column and then from the ten's column as shown below.

hundreds	tens	units		hundreds	tens	units		hundreds	tens	units
a	b	c	☛	(a – 1)	(b + 10)	c	☛	(a – 1)	(b + 9)	(c + 10)
– c	b	a		c	b	a		c	b	a
								(a – 1 – c)	(b + 9 – b)	(c + 10 – a)

Writing this difference in expanded notation we get: $100(a - 1 - c) + 10(b + 9 - b) + (c + 10 - a) = 99a - 99c = 9(11a - 11c)$, which is a multiple of 9. The sum of these digits, $(a - 1 - c) + (b + 9 - b) + (c + 10 - a) = 18$, is also a multiple of 9. Note that if either of these digits is reported, the other can be predicted. Also note that if these digits, 1 and 8, are added, the result is 9, a fact used in variation #2 of the trick.

You may wish to verify other cases for this trick. For example, suppose *abc* and permutation *bac* were chosen, where $b > a$. Show that the difference (*bac* – *abc*) is a multiple of 9 and that the sum of the digits of the difference is also a multiple of 9.

Hints to Further Investigations

1. The computations are shown for a choice of 3874 and the permutation 7348.

   7348 – 3874 = 3474 and 3 + 4 + 7 + 4 = 18

   Note that the answer, 18, is a multiple of 9, and if one of the digits is known, the other will also be known.

2. Suppose $a > b$. See table next page.

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	tens	units
	a	b
–	b	a

☛

tens	units
(a – 1)	(b + 10)
b	a
(a – 1 – b)	(b + 10 – a)

Note that the sum of the two digits in the difference, $(a - 1 - b) + (b + 10 - a)$, is 9. So if one digit is known, you can subtract this from 9 to find the other digit.

3. The computations are shown for a choice of 21; $21^2 - 12^2 = 297$ and $2 + 9 + 7 = 18$.

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THE TRICK: Magical Number Nine

Variation One

1. Select a three-digit number in which none of the three digits is the same.
2. Form another three-digit number by rearranging the original digits.
3. Now subtract the two three-digit numbers and add the digits in this difference.
4. From the first (or last) digit in the sum, you can tell the other digit.

Variation Two

1. Follow the steps above through step #3.
2. If the sum is not one digit, continue adding the digits in the result until you reach a one digit number. You can predict this digit.

UNRAVELING THE TRICK

1. Choose three different variables for the hundred's, ten's and unit's digit of the number, and write them in appropriately labelled columns.
2. Borrow one from the hundred's column, and rewrite the hundred's and ten's digits.
3. Borrow one from the ten's column, and rewrite the ten's and unit's digits.
4. Suppose the permutation selected is the original digits in reverse order. Subtract these digits from those appearing in the table.
5. Sum the digits in the difference above, and use this result to explain the trick.

QUESTIONS TO PONDER

1. What would happen if all the digits in the chosen number were the same?
2. Can two of the digits be the same?
3. Can you verify the trick if one uses a permutation other than the original digits reversed?
4. Will the sum of the digits in the difference always be a multiple of 9?

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FURTHER INVESTIGATIONS

1. Show with numerical illustrations that the two variations of the trick are not limited to three-digit numbers.

2. Using algebra, verify the following variation of this trick.

   Think of a two-digit number with unequal digits. Reverse the digits and find the difference of the two numbers. Tell me one of the digits in the difference and I will tell you the other.

3. Show with numerical illustrations that the following variation of the trick works.

   Take a number and transpose the digits. Square each of these numbers and compute the difference. Now, add the digits in the difference, and give me the first digit in the sum. I will tell you the last digit in the sum.

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NUMBER 15 THE MISSING DIGIT

Here is another trick based on special properties of the number 9.

PERFORMING THE TRICK

1. Choose a volunteer and have her/him pick a four-digit number, then give her/him directions a–d below.
   a. Compute the sum of the digits in the four-digit number you picked.
   b. Cross-out any one of the digits in your original number and write the remaining three digits as a three-digit number.
   c. From this three-digit number subtract the sum of the digits previously computed.
   d. Sum the digits in this difference, and announce the answer.
2. From this answer, you can reveal the digit that was crossed out in the original number.

HOW IT WORKS

When the result is announced, subtract it from the smallest multiple of 9 which is greater than the result. This difference will be the missing digit.

AN EXAMPLE

Selected four-digit number	3672
Sum of the digits	3 + 6 + 7 + 2 = 18
Cross-out a digit	36~~7~~2
Form a three-digit number	362
Subtract the sum of digits	− 18
	344
Sum these digits	3 + 4 + 4 = 11
Reveal the answer	11

At this point, you can predict that the 7 was crossed out, since 18 – 11 = 7.

WHY IT WORKS

First it can be shown that the difference of any number and the sum of the digits of that number is always a multiple of 9.

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To explain the trick, suppose we choose a four-digit number $N = abcd$. Written in expanded notation, we have:

$$N = 1000a + 100b + 10c + d.$$

The sum of the digits of N is $S = (a + b + c + d)$.

$$\begin{aligned} N - S &= (1000a + 100b + 10c + d) - (a + b + c + d) \\ &= 999a + 99b + 9c \\ &= 9(111a + 11b + c) \end{aligned}$$

The result is a multiple of 9.

If we delete the c in N, then the computations below follow.

three-digit number without c	$100a + 10b + d$	
subtract the sum of digits	$100a + 10b + d - (a + b + c + d)$	$= 99a + 9b - c$
		$= 9(11a + b) - c$

The result is a multiple of 9 minus c. When this number is reported, you need only determine what one-digit number must be added to the result to produce a multiple of 9. The number you add will be the same as the missing digit.

The argument will be similar no matter which of the digits (a, b, c, or d) is crossed out. There is just one problem with this argument. If a 0 or a 9 is crossed out in N, then $N - S$ will be a multiple of 9 and you will not be able to tell if the missing digit is a 0 or a 9. For example, consider crossing out the 0 in $N = 4087$. We find $S = 4 + 0 + 8 + 7 = 19$ and $487 - 19 = 468$ and $4 + 6 + 8 = 18$, a multiple of 9.

Hints to Further Investigations

1. $N - S = (100a + 10c + d) - (a + b + c + d) = (99a + 9c) - b$. To produce a multiple of 9, you must add the digit b.
2. You might suggest deleting any non-zero digit in the four-digit number.
3. If the digit 4 is deleted from the six-digit number 387241, then $N - S = 38721 - (3 + 8 + 7 + 2 + 4 + 1) = 38696$ and $3 + 8 + 6 + 9 + 6 = 32$. You must add 4 to 32 to get the next multiple of 9.
4. Assume $a > c$. Subtract as shown. Note that the sum of the first and last digits is 9.

	a	b	c		(a – 1)	(b + 9)	(c + 10)
–	c	b	a		c	b	a
					(a – 1 – c)	9	(c + 10 – a)

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5. Since 457 leaves a remainder of 7, and 47 leaves a remainder of 2, the deleted digit was 7 – 2 = 5. Note that 590 leaves a remainder of 5, as does 50 after the 9 is deleted. Could you tell if a 0 or a 9 was deleted?

6. For a four-digit number (with digits: a, b, c, and d),

$$\begin{aligned} & 1000a + 100b + 10c + d + 8(a + b + c + d) \\ = \; & 1008a + 108b + 18c + 9d \\ = \; & 9(112a + 12b + 2c + d). \end{aligned}$$

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THE TRICK: The Missing Digit

1. Choose a four-digit number and compute the sum of its digits.
2. Cross out any one of the digits in the original number, and write the remaining three digits as a three-digit number.
3. From this three-digit number, subtract the sum of the digits previously computed.
4. Sum the digits in this difference.
5. From this result you can tell the digit that was crossed out in the original number.

UNRAVELING THE TRICK

1. Choose four different variables to represent the thousand's, hundred's, ten's, and unit's digits of a four-digit number, N.

2. Let S be the sum of the four digits in N.

3. Use algebra to show that $(N - S)$ is a multiple of 9.

4. Delete one digit from the four-digit number and express the remaining three digits as a three-digit number in expanded notation. Label this number M.

5. Find $(M - S)$ and simplify the result.

6. Explain the trick by using the observation from step #3 and considering what number must be added to the expression in step #5 to produce a multiple of 9.

QUESTIONS TO PONDER

1. Are there any numerical restrictions on the four variables chosen? Could they all be the same number?

2. Would the trick work if a 0 were deleted from the original number? If a 9 were deleted? Why or why not?

3. How could the trick be altered to take the results of Question #2 into account?

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FURTHER INVESTIGATIONS

1. Verify the argument in *Unraveling the Trick* if a different digit from the one you chose previously is deleted from the number, *N*.

2. How could you alter this trick so that you could tell if someone deleted a 0 or 9 and also tell which of the two digits was deleted?

3. Provide numerical examples to show that this trick works with any three-digit number and with any six-digit number.

4. Use algebra to verify that the following variation of the trick always works.

   Ask a person to write down three figures, then to reverse the figures and subtract. If he tells you only the left-end figure in the remainder, you can tell the whole remainder. The central figure in the remainder is always 9, and the two end figures always sum to 9. Hence, having the left-end figure (say 2), the whole remainder would be known (297).

5. Give numerical illustrations of the following trick and determine what happens if 0 or 9 is deleted from the chosen number.

   Think of a number of three or more digits, divide by 9, and name the remainder. Erase one figure from the original number, divide by 9, and tell me this remainder. Now I will tell you what figure you erased. The trick: If the second remainder is less than the first, the figure erased is the difference between the remainders. If the second remainder is greater than the first, the figure equals 9 minus the difference of the remainders.

6. Use algebra to verify the following trick. Write down any number. Find the sum of its digits, multiply by 8, and add the result to the original number. The result is a multiple of 9, that is, reduces to 9 by casting out nines.

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NUMBER 16 A PERMUTATION TRICK

This is a trick that I devised based on permutations.

PERFORMING THE TRICK

1. Choose a volunteer and give her/him three blank cards or slips of paper, then give the instructions in a–g below.
   a. Take the three blank cards and write any single digit on the first card, any single digit on the second card (please do not make both digits zeros), and a decimal point on the third card.
   b. Without revealing the numbers chosen, find all possible decimal numbers that can be formed by rearranging the three cards. There should be six.
   c. Sum the six numbers.
   d. Multiply this sum by 100.
   e. Divide this result by 11.
   f. Divide this result by 3.
   g. Divide this result by the sum of the two digits that was first selected.
2. Announce that you can predict the answer.

HOW IT WORKS

The answer is always 37.

AN EXAMPLE

If you chose 5 and 3 as your digits, the computations below and on the next page follow.

The six possible numbers formed and added are:

$$\begin{array}{r} 53. \\ 35. \\ 5.3 \\ 3.5 \\ .53 \\ +\ .35 \\ \hline \text{Sum} = 97.68 \end{array}$$

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Multiply the sum by 100	$100 \times 97.68 = 9768$
Divide the result by 11	$9768 \div 11 = 888$
Divide the result by 3	$888 \div 3 = 296$
Divide the result by the sum of the two numbers first selected	$296 \div (5 + 3) = 37$

WHY IT WORKS

Represent your two digits by x and y. The six possible arrangements are shown as they would be written in decimal notation.

$xy.$ $yx.$ $x.y$ $y.x$ $.xy$ $.yx$

Summing these numbers and writing the sum in expanded notation gives the following:

$$(x + y) \times 10 + (2x + 2y) \times 1 + (2x + 2y) \times 1/10 + (x + y) \times 1/100.$$

After factoring a 2 from the expression $(2x + 2y)$, we have:

$$(x + y) \times 10 + (x + y) \times 2 + (x + y) \times 2/10 + (x + y) \times 1/100.$$

Note that $(x + y)$ is a common factor in the four terms of this sum. Factoring $(x + y)$ we have:

$$(x + y)(10 + 2 + 2/10 + 1/100) = (12.21)(x + y).$$

Now, according to the directions in the problem we compute as follows.

$$\begin{aligned} 12.21(x + y) \times 100 &= 1221(x + y) \\ 1221(x + y) \div 11 &= 111(x + y) \\ 111(x + y) \div 3 &= 37(x + y) \\ 37(x + y) \div (x + y) &= 37. \end{aligned}$$

Of course this fails if x and y are both zero.

Hints to Further Investigations

1. There are 24 permutations. The sum in expanded notation is $100(2x + 2y) + 10(4x + 4y) + (6x + 6y) + 1/10(6x + 6y) + 1/100(4x + 4y) + 1/1000(2x + 2y) = 246.642(x + y)$. What whole numbers divide 246642?
2. The following set of instructions lead to an answer of 0: multiply by 100, divide by 11, divide by 3, divide by 37, and from this result subtract the sum of the two numbers on the cards.
3. There are 24 permutations and the sum is $100(2x + 2y + 2z) + 10(4x + 4y + 4z) + (6x + 6y + 6z) + 1/10(6x + 6y + 6z) + 1/100(4x + 4y + 4z) + 1/1000(2x + 2y + 2z)$. Analyze the result as was done in the original trick.
4. Show that $100x + 10x + x$ divided by $3x$ is 37.

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THE TRICK: A Permutation Trick

1. Take three blank cards and write any single digit on the first card, any single digit on the second card (please do not make both digits zeros), and a decimal point on the third card.
2. Without revealing the numbers chosen, find all possible decimal numbers that can be formed by rearranging the three cards. There should be six.
3. Sum the six numbers.
4. Multiply this sum by 100.
5. Divide this result by 11.
6. Divide this result by 3.
7. Finally, divide this result by the sum of the two digits that were first selected.
8. You can predict the answer.

UNRAVELING THE TRICK

1. Choose two different variables to represent the two digits on the cards.

2. Write all the permutations of the three cards.

3. Sum the permutations and express the sum in expanded notation.

4. Multiply the sum by 100 and express the result in factored form.

5. Explain the trick by showing that the algebraic expression from step #4 is evenly divisible by 11, 3, and the sum of the two digits.

QUESTIONS TO PONDER

1. Are there any numerical restrictions on the variables? Could they be the same number? Could one, or the other, or both be zero?

2. Why was the sum multiplied by 100? Could you predict the answer if this step was omitted?

3. How could you alter the computational directions in the trick to predict the sum of the digits of the chosen numbers?

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FURTHER INVESTIGATIONS

1. Construct a variation of this trick using the following four cards: two cards with single digits of one's choice, one card with a zero, and one card with a decimal point. How would you predict the answer to this trick?

2. Think of a new set of directions to be used (after the numbers have been summed) in the original trick. In this case how would you predict the answer?

3. Construct a trick using four cards: three with single digits of one's choice and one with a decimal point. How would you predict the answer to this trick?

4. Analyze the following trick which also has 37 as the predicted answer.

   Choose any three-digit number with three identical digits. Divide the number by the sum of the digits. The result will always be 37.

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NUMBER 17 THE REAPPEARING NUMBER

The following is an amazing trick using division by prime numbers.

PERFORMING THE TRICK

1. Choose a volunteer and ask her/him to write down any two-digit prime number greater than 13.
2. Have her/him attach a zero to this number and form a six-digit number by repeating these three digits.
3. Have the person pass the paper to five other people. Have each of them do successive divisions by the primes: 2, 5, 7, 11, and 13. The first person divides the six-digit number by 2, the second person divides the resulting quotient by 5, the third person divides the second quotient by 7, and so on. In all the divisions the remainders are zeros. When the final division is completed, the last quotient is written on the paper and passed back to the original person who thought of the number.
4. The person will recognize this two-digit prime number.

HOW IT WORKS

No matter what two-digit prime number is selected, the divisions by 2, 5, 7, 11, and 13 will produce a quotient which is the original two-digit number.

AN EXAMPLE

If the subject chose 31 as the prime, the six-digit number is 310310. The results of the five divisions are shown below.

$$2\overline{)310310}\quad 155155$$

$$5\overline{)155155}\quad 31031$$

$$7\overline{)31031}\quad 4433$$

$$11\overline{)4433}\quad 403$$

$$13\overline{)403}\quad 31$$

WHY IT WORKS

Let *ab* represent the two-digit prime number selected. The six-digit number is then written *ab*0*ab*0. Call this number *P* and express it in expanded notation.

Now, $P = 100000a + 10000b + 1000(0) + 100a + 10b + 0$

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$= 100100a + 10010b$

$= 10(10010a + 1001b)$.

Let $Q = \frac{P}{2} = \frac{10(10010a + 1001b)}{2} = 5(10010a + 1001b)$ which is the quotient after the division by 2.

Let $R = \frac{Q}{5} = \frac{5(10010a + 1001b)}{5} = 10010a + 1001b$ which is the quotient after division by 5.

Let $S = \frac{R}{7} = \frac{10010a + 1001b}{7} = \frac{10010a}{7} + \frac{1001b}{7} = 1430a + 143b$ which is the quotient after division by 7.

Let $T = \frac{S}{11} = \frac{1430a + 143b}{11} = \frac{1430a}{11} + \frac{143b}{11} = 130a + 13b$ which is the quotient after division by 11.

Finally let $U = \frac{T}{13} = \frac{130a + 13b}{13} = \frac{130a}{13} + \frac{13b}{13} = 10a + b$ which is the quotient after division by 13.

Note that this result is the original two-digit prime number written in expanded notation. Also, note that this trick does not depend upon the choice of *a* or *b*.

Hints to Further Investigations

1. No, they could be any two digits.
2. The six-digit number *abcabc* in expanded notation is $100000a + 10000b + 1000c + 100a + 10b + c$ which simplifies to $100100a + 10010b + 1001c$. Show that each of these numerical coefficients is divisible by 7, 11, and 13.
3. In the example given, the result is 1111. In general if the digits are *abcd*, the sum of the four numbers formed in expanded notation is $1000(a + b + c + d) + 100(a + b + c + d) + 10(a + b + c + d) + (a + b + c + d)$.

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THE TRICK: The Reappearing Number

1. Write down any two-digit prime number greater than 13.
2. Attach a zero to this number and form a six-digit number by repeating these three digits.
3. Pass the paper to five other people. Have each of them do successive divisions by the primes: 2, 5, 7, 11, and 13. The first person divides the six-digit number by 2, the second person divides the resulting quotient by 5, the third person divides the second quotient by 7, and so on. In all the divisions the remainders are zeros. When the final division is completed, the last quotient is written on the paper and passed back to the person who thought of the number.
4. When you get the paper back, you will recognize the two-digit prime number on the paper.

UNRAVELING THE TRICK

1. Choose two different variables to represent the ten's and unit's digits of the chosen prime.

2. Represent the six-digit number formed from the chosen prime in expanded notation, simplify this expression, and then write it in factored form.

3. Divide the expression from step #2 successively by 2, 5, 7, 11, and 13 simplifying the quotient after each division.

4. Use the result of step #3 to explain the trick.

QUESTIONS TO PONDER

1. Are there any numerical restrictions on the variables? Must the two digits form a prime number for the trick to work?

2. If the first division performed in the trick was preceded by division of the chosen prime, what would happen to the trick?

3. Does the order in which the divisions are performed affect the outcome of the trick?

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FURTHER INVESTIGATIONS

1. Do the two digits selected by the subject have to form a prime number for the trick to work?

2. Suppose the subject was asked to write down a three-digit number and repeat these three digits to form a six-digit number. Alter the trick using successive divisions by 7, 11, and 13. Use algebra to verify that this version of the trick always results in the three-digit number selected.

3. Have someone choose a four-digit number using four different digits. Form three other numbers by moving the thousand's digit to the unit's place and all other digits one place to the left; for example, 3785, 7853, 8537, and 5378. Sum these four numbers and divide the result by the sum of the four original digits chosen. What is the result? Analyze this trick algebraically.

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NUMBER 18 THINK OF A NUMBER

One of the oldest and most used number tricks is "think of a number." After leading your subject through a series of arithmetic computations, you predict the number thought of by your subject. Early texts dealing with number games refer to the process of predicting someone's number as *divining a number.* Here is one example of "divining a number."

PERFORMING THE TRICK

1. Ask a volunteer to think of a number then perform the computations in a–e below.
   a. Multiply the number by 2.
   b. Add 6 to the product.
   c. Add this result to the square of the original number.
   d. Subtract 5 from this sum.
   e. Take the square root of the difference and announce the result.
2. Here you can reveal the original number.

HOW IT WORKS

In this version of the trick, you merely subtract one from the number reported by the subject.

AN EXAMPLE

If the subject chose 124, the following computations are done by the subject.

124

$2 \times 124 = 248$

$248 + 6 = 254$

$254 + (124)^2 = 15{,}630$

$15{,}630 - 5 = 15{,}625$

$\sqrt{15625} = 125$

When 125 is reported, your prediction is 124.

WHY IT WORKS

Let x equal the number selected by the subject. The algebraic computations for the trick are shown on the next page.

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Think of a number:	x
Multiply the number by 2:	$2x$
Add 6 to the product:	$2x + 6$
Add the square of the original number:	$2x + 6 + x^2 = x^2 + 2x + 6$
Subtract 5 from this sum:	$x^2 + 2x + 6 - 5 = x^2 + 2x + 1 = (x + 1)^2$
Take the square root of the difference:	$\sqrt{(x + 1)^2} = x + 1$

Thus, when 1 is subtracted from the answer reported ($x + 1$) you arrive at the original number thought of by the subject, namely, x. If the number selected is large, your subject may need a calculator to complete the computations.

In order to use this trick repeatedly, it is important that the prediction not always be 1 less than the number reported. To hide this fact, you may stop earlier in the sequence (say after 5 has been subtracted from the sum) and complete the final computations yourself (i.e., taking the square root and subtracting 1).

To illustrate this, consider this variation of *divining a number.*

> Think of a number, subtract 1 from this number, double your result, subtract 1, add the original number, and tell me the result.

The algebra goes as follows:

x

$x - 1$

$2(x - 1) = 2x - 2$

$(2x - 2) - 1 = 2x - 3$

$2x - 3 + x = 3x - 3$

When this result is reported to you, add 3 and divide by 3 to *divine* the original number selected.

Hints to Further Investigations

1. (1) Think of x and the result is $10x + 3$.

   (2) Think of x and the result is $100x + 320$.

   (3) Think of x and the result is $2x - 1$.

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THE TRICK: Think of a Number

1. Have someone think of a number.
2. Ask them to perform computations a–e below.
   a. Multiply the number by 2.
   b. Add 6 to the product.
   c. Add this result to the square of the original number.
   d. Subtract 5 from this sum.
   e. Take the square root of the difference.
2. Here you can predict the original number selected.

UNRAVELING THE TRICK

1. Choose a variable to represent the selected number.

2. Write algebraic expressions for the number after each computational step in the trick.

3. Simplify the last algebraic expression and subtract 1 from it.

4. Explain the trick by comparing the result of step #3 with the original number chosen.

QUESTIONS TO PONDER

1. Does the number chosen have to be positive for the trick to work? Try starting with -5. Try starting with 0.

2. Are there numerical restrictions on the number chosen? Can you choose any positive rational number?

3. Is $\sqrt{(x+1)^2} = x + 1$ always true? Try $x = -3$.

4. How could the directions to the trick be altered to take the observations in #1–3 above into account?

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FURTHER INVESTIGATIONS

1. Three number games are shown below. Show why these tricks always work.

   (1) Think of a number, say 6

Multiply it by 3	$6 \times 3 =$	18
Add 1	$18 + 1 =$	19
Multiply by 3	$19 \times 3 =$	57
Add to this the original number	$57 + 6 =$	63 Report the result.

   The resulting number will always end with 3. Delete the 3. What remains is the original number (6).

   (2) Think of a number, say 6

Double it	$6 \times 2 =$	12
Add 4	$12 + 4 =$	16
Multiply by 5	$16 \times 5 =$	80
Add 12	$80 + 12 =$	92
Multiply by 10	$92 \times 10 =$	920 Report the result.

   Subtract 320 from the result (920 – 320 = 600). Delete the two zeros. This gives the original number (6).

   (3) Think of a number, say 6

Multiply this number by itself	$6 \times 6 =$	36
Take 1 from the original number	$6 - 1 =$	5
Multiply this number by itself	$5 \times 5 =$	25
Find the difference between this product and the previous product	$36 - 25 =$	11 Report the result.

   Add 1 to the number reported (11 + 1 = 12) and halve this number (12 ÷ 2) to find the original number (6).

2. Any algebraic identity may be used as the basis of a number game. For example, the equation, $\sqrt{\frac{(5x)^2 + (12x)^2}{x^2}} = 13$, leads to the following game.

   Think of a number; first multiply your number by 5, then square the result. Now, multiply your original number by 12, and square the result. Add these two squares together, and divide the sum by the square of your original number. Now, compute the square root of this value (use a calculator, if necessary). Don't tell me the answer for I already know it. It is 13.

   Devise a trick for the equation: $x^2 + 2x + 1 = (x + 1)^2$.

3. Devise a trick for the equation: $\frac{\sqrt{(3x)^2 + (4x)^2} + 15}{5} = x + 3$

4. Find other algebraic identities and write *divining* tricks based on them.

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NUMBER 19 DIVINING FROM A TABLE

Although we may associate binary arithmetic with the computer age, this trick is based on binary notation and was published in 1857. (*The Magician's Own Book*, New York: Dick and Fitzgerald, Publishers, 1857. p. 241)

PERFORMING THE TRICK

1. Choose a volunteer to think of a number from 1 to 60. (You might suggest thinking of one's age or anything that you are sure will not exceed 60.)
2. Give your subject the six cards shown on the next page.
3. Ask for all the cards containing her/his number.
4. Immediately, you can reveal her/his number.

HOW IT WORKS

As you are handed the cards which contain the subject's number, add the numbers found in the upper right-hand corner of each card. This sum will match your subject's choice.

AN EXAMPLE

If the subject selects the number 50, three cards which contain the number 50 will be returned to you. Adding the numbers in the upper right corners of these three cards, you find the sum 2 + 16 + 32 = 50.

WHY IT WORKS

Distribution of the numbers 1 to 60 on the six cards is based upon the binary coding of each number. All numbers that have a 1 in the unit's place of the binary code appear on the "1" card. All numbers that have a 1 in the two's place (second place from the right) of the binary code appear on the "2" card. All numbers that have a 1 in the four's place (third from the right) in the binary code appear on the "4" card, etc. For example, the binary code for 50 is 110010, which in expanded notation is $1 \times 32 + 1 \times 16 + 0 \times 8 + 0 \times 4 + 1 \times 2 + 0 \times 1 = 50$. Thus, 50 appears on the "2" card, the "16" card, and on the "32" card. When these cards are returned to you, you need only add the values in the upper right corner, 2 + 16 + 32 = 50, to determine the chosen number. In general, any number *x* from 1 to 60 has a unique binary code consisting of a string of 0s and 1s no longer than six digits. When this binary code is written in expanded notation, *x* will be the sum of a unique combination of the following set of values: 1, 2, 4, 8, 16, 32. Another way of saying this is that *x* will appear on a unique combination of the six binary cards shown on the next page. Thus, summing the values in the upper right corners (the powers of two) will produce the value *x*.

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3	5	7	9	11	1
13	15	17	19	21	23
25	27	29	31	33	35
37	39	41	43	45	47
49	51	53	55	57	59

5	6	7	13	12	4
14	15	20	21	22	23
28	29	30	31	36	37
52	38	39	44	45	46
47	53	54	55	60	13

9	10	11	12	13	8
14	15	24	25	26	27
28	29	30	31	40	41
42	43	44	45	46	47
56	57	58	59	60	13

3	6	7	10	11	2
14	15	18	19	22	23
26	27	30	31	34	35
38	39	42	43	46	47
50	51	54	55	58	59

17	18	19	20	21	16
22	23	24	25	26	27
28	29	30	31	48	49
50	51	52	53	54	55
56	57	58	59	30	60

33	34	35	36	37	32
38	39	40	41	42	43
44	45	46	47	48	49
50	51	52	53	54	55
56	57	58	59	60	41

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Hints to Further Investigations

1. Since $2^8 = 256$ and $2^9 = 512$, it would take nine cards.
2. The base two numeral 11111111 is 255 in base ten. Note that 255 is equal to $2^8 - 1 = 256 - 1$.

3/4. Since the division, $50 \div 2 = 25$, leaves a zero remainder, there is a 0 in the unit's place in the binary code for 50. Since $25 \div 2 = 12$ with a remainder of 1, the "twenty-five groups of two" can be regrouped into "twelve groups of four" with one "2" left over. This 2 is represented by a 1 in the two's column of the binary code. Follow this reasoning as you continue the divisions by 2.

5. The first twenty counting numbers appear in their decimal and ternary codes in the table below.

Decimal	Ternary
1	1
2	2
3	10
4	11
5	12
6	20
7	21

Decimal	Ternary
8	22
9	100
10	101
11	102
12	110
13	111
14	112

Decimal	Ternary
15	120
16	121
17	122
18	200
19	201
20	202

Since each place value in the ternary system may use the digits 0, 1, or 2, it is necessary to identify which numbers have a 1 and which a 2 in any place value location. Consider the following scheme for making cards where two cards are used for each place value location in the ternary code.

Place Value	Card Number	First 20 Counting Numbers Appearing on the Cards
units	1	1, 4, 7, 10, 13, 16, 19
units	2	2, 5, 8, 11, 14, 17, 20
threes	3	3, 4, 5, 12, 13, 14
threes	4	6, 7, 8, 15, 16, 17
nines	5	9, 10, 11, 12, 13, 14, 15, 16, 17
nines	6	18, 19, 20

The first numbers appearing on the cards above would act as a key as was done in the original trick. The numbers 1–20 can be predicted using these six cards.

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THE TRICK: Divining from a Table

1. Have someone think of a number from 1 to 60.
2. Give them the six cards.
3. Ask for all the cards containing her/his number.
4. From the cards returned to you, you can predict her/his number.

UNRAVELING THE TRICK

1. Verify that all the numbers on the "1 card" (the card with a "1" in the upper right corner) have a 1 in the unit's place of their binary codes.

2. Verify that all numbers on the "2 card" have a 1 in the two's place of their binary codes.

3. Verify that all the numbers on the "4," "8," "16," and "32 cards" have 1s in the four's, eight's, sixteen's, and thirty-two's place (respectively) of their binary codes.

4. Use the observations above to explain why any number (from 1 to 60) written in base ten can be predicted by adding the numbers in the upper right corner of each card on which that given number appears.

QUESTIONS TO PONDER

1. With reference to the binary system, what is the significance of the set of numbers found in the upper right corners of the six cards?

2. Why is the binary system especially well-suited for this trick?

3. What is the sum of the six numbers found in the upper right corners of the cards? What other prediction numbers could be listed on the six cards?

4. If a seventh card was constructed, what number would you place in the upper right corner? What is the largest number that could be predicted using a set of seven binary cards?

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FURTHER INVESTIGATIONS

1. How many binary cards would be needed to allow a subject to think of a number up to 500?

2. If you constructed 8 binary cards, how large a number could you predict? Suppose you had *n* binary cards?

3. The following computations illustrate an algorithm for changing the decimal number, 50, to its binary code, 110010.

Computation	Quotient	Remainder	Binary Coding
50 ÷ 2 =	25	0	unit's place
25 ÷ 2 =	12	1	two's place
12 ÷ 2 =	6	0	four's place
6 ÷ 2 =	3	0	eight's place
3 ÷ 2 =	1	1	sixteen's place
1 ÷ 2 =	0	1	thirty-two's place

Thus, 50 in base ten is 110010 in base two.

Use this algorithm to find the binary code for other numbers between 1 and 60. Verify your coding by checking the six binary cards. If you have a scientific calculator which converts base ten to base two, use this to verify your codes.

4. Explain why the algorithm above works.

5. Devise a set of cards based on ternary coding (base three notation) that could be used to predict a number.

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NUMBER 20 THE DOMINO CHAIN

The following is a well-known trick using a double-six (28 piece) domino set. It is the only activity in this book that resorts to physical deception.

PERFORMING THE TRICK

1. Mix the set of dominoes and place them face up on a table.
2. Ask a volunteer to form a single chain using all the dominoes and matching them end to end as in normal play.
3. After the chain is completed, have her/him remove your prediction from a sealed envelope and read it.
4. Your prediction will match the beginning and end numbers in the domino chain.

   This trick may be repeated any number of times. Although your prediction may change, it will match the ends of the chain.

HOW IT WORKS

Here is the trickery. Before you mix and place the dominoes face up on the table, remove and conceal one piece. Make sure the domino you conceal is not a double. Your prediction will consist of the two numbers on this hidden domino. Leave your prediction sealed in an envelope in plain sight of your subject. When your subject has completed the chain using all of the dominoes on the table, the first and last numbers in the chain will match your prediction. If the trick is to be repeated, secretly return the piece to the set and remove a different piece. The numbers on this new piece will be your next prediction.

AN EXAMPLE

As you set up the trick, suppose you have removed the 3-5 domino. In an envelope, you have sealed your prediction "3 and 5." No matter how your subject begins the chain, to use all the pieces on the table, the completed chain will begin and end with the numbers 3 and 5.

WHY IT WORKS

This trick is based on the fact that in any complete chain of a 28-piece domino set the two end numbers will always match. This is not true of all sets of dominoes, so use a set of 28 pieces.

Suppose you have removed the domino with numbers x and y, where x is not equal to y.

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Since you are holding the x-y domino, the beginning and end of the subject's chain of 27 dominoes must match the x and y in your prediction. The domino you hold will complete the matching ends in a 28-piece chain.

Hints to Further Investigations

1. Starting with a 3-3 domino one might reason as follows. If a 3-3 domino starts the chain, it must be followed with a 3-x domino (with $x \neq 3$). This leaves five dominoes on which a 3 appears on only one end. But, as they are used in the chain, they must be paired, which leaves one 3-y domino (with $y \neq 3$) that is not paired with a 3. This remaining domino must be paired with a y and this leaves the 3 on the other end of the chain.
2. If you conceal a double, say 3-3, this leaves six dominoes with a 3 on only one end. These six can be paired in the completed chain so the chain could end with a number that is not a 3.
4. To verify this, it is useful to know the formula for summing the first n counting numbers:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Now, consider the set of double-six dominoes arranged as follows:

0-0						
0-1	1-1					
0-2	1-2	2-2				
0-3	1-3	2-3	3-3			
0-4	1-4	2-4	3-4	4-4		
0-5	1-5	2-5	3-5	4-5	5-5	
0-6	1-6	2-6	3-6	4-6	5-6	6-6

The number of dominoes in the seven columns is: $1 + 2 + 3 + 4 + 5 + 6 + 7$.

Using the formula for summing consecutive counting numbers we conclude:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = \frac{7(7+1)}{2} = \frac{(6+1)(6+2)}{2} = \frac{56}{2} = 28.$$

This argument can be generalized for a double-n set of dominoes.

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THE TRICK: The Domino Chain

1. Mix the 28-piece set of dominoes and place them face up on a table.
2. Ask your subject to form a single chain using all the dominoes and matching them end to end as in normal play.
3. You can predict the beginning and ending numbers in the domino chain.

UNRAVELING THE TRICK

This is the only trick in this book which actually uses sleight of hand. One domino (not a double) is concealed. The numbers on this domino become your prediction.

FURTHER INVESTIGATIONS

1. Use a 28-piece set of dominoes to experiment with chains. Start a chain with a 3-3 domino and see if you can end with a number other than 3. Start with a 3-x domino (where x is not 3) and see if you can end with a number other than 3. Write an explanation of why any chain that begins with x must end with x.

2. Explain why the trick does not work if you conceal a double domino.

3. A double-five set of dominoes (from 0-0 to 5-5) has 21 pieces. Experiment with such a set to show that a complete chain of the 21 pieces does not have to begin and end with the same number. Can you form a complete chain using all 21 pieces?

4. Show that the number of dominoes in a set going from a 0-0 domino to an n-n domino is $\frac{[(n+1)(n+2)]}{2}$.

5. There are approximately 7×10^{12} ways that a 28-piece set of dominoes can be arranged in a chain using the rules of the game. A double-five set of dominoes (21 pieces) cannot be put into a continuous chain using all 21 pieces (see #1 above). If you removed the 0-2 and the 1-3 dominoes from a double-five set, however, you can form a complete chain with the remaining 19 pieces. Try it.

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